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PLANETARY THEORY

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PLANETARY THEORY

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PREFACE

THE chief purpose of this volume is the development of methods for the calculation of the general orbit of a planet. If an accuracy comparable with that of modern observation is to be attained in any particular case, the choice of the method to be adopted may be an important factor in the amount of calculation to be performed. Not only should the general plan of procedure be efficient, but full consideration of the details of the work should be given in advance. We have attempted to anticipate the difficulties which arise, not only in the older methods but also in those developed here, by setting forth the various devices which may be utilised when needed.

While the developments given below are intended to be complete in the sense that they should not require a knowledge of the subject drawn from other sources, the volume is not supposed to be a substitute for an extended treatise like that of Tisserand. It contains, for example, no detailed account of such classical theories as those of Leverrier, Hansen and Newcomb. It does, however, attempt to indicate that most of the methods previously used ultimately reduce to two. One of the methods involves a change of the variables to elliptic elements, while the other consists of a direct calculation of expressions for the coördinates. An example of each of these general plans is given and worked out in detail.

Few references to previous work have been made and those furnished are merely incidental. It has seemed unnecessary to repeat material which the student can find equally well in Tisserand's treatise or in vol. IV of the *Encyklopädie der Mathematischen Wissenschaften*. While a critical estimate of the merits and demerits of previous works would doubtless be of assistance to anyone planning to carry out detailed calculations for the theory of a particular planet, in the past the methods which have been adopted have been sometimes chosen less on account of their efficiency than for other reasons, and the same will probably be to some extent true in the future. Nevertheless it

is still advisable to give consideration to each of these plans, and we have attempted, by occasional remarks, to aid the student in this respect.

The mathematical processes which are used in developing the theories of the planets and satellites from the laws of motion are largely formal. While mathematical rigour is desirable when it can be attained, nearly all progress in the knowledge of the effects of these laws would be stopped if complete justification of every step in the process were demanded. The use of formal processes is justified whenever experience shows that the results, not otherwise obtainable, are useful for the prediction of physical phenomena. Thus when calculating with an infinite series whose convergence properties are not known, one has to be guided by the results obtained; if the series appears to be converging with sufficient rapidity to yield the needed degree of accuracy, there is no choice save that of using the numerical values which it gives. We have not attempted to deal with convergence questions, but have retained throughout the practical point of view mentioned in the first sentence of this preface.

Considerable portions of the volume are new in the sense that if they had not been given here, they would have been printed in abbreviated form in the current journals. In particular is this true of the last two chapters. The novelty, however, consists mainly in the adaptation and further exploitation of previously known devices. Some of these extensions owe their effectiveness to a recent publication of tables of certain functions*, or to the introduction of mechanical computing aids. An example is the attention given to development by harmonic analysis.

The following sentences give a brief summary of the contents of the volume. In the first chapter, various forms of the equations of motion are derived, other possible forms being suggested. The second chapter is a collection of various expansion theorems which are or may be needed in the later developments. A short account of the essential properties of elliptic motion follows. Various methods for developing the disturbing function and disturbing forces are set forth in Chapter IV. Chapter V contains

* See footnote, p. 182.

the elements of the theory of canonical variables so far as it is needed in the later work. This theory is usually difficult for a student to grasp, and we have tried to simplify the exposition so that he may not only be able to understand it but also to make use of it as a tool for investigation. In Chapter VI, it is shown how this theory may be efficiently applied to the calculation of the orbit of a planet; the basis of the method is the use of the transformation to eliminate the short-period terms as a first step, leaving the long-period and secular terms to be dealt with separately.

In Chapter VII, the direct calculation of the coordinates with the use of the true orbital longitude as the independent variable is developed with sufficient detail for the formation of an approximate or of an accurate theory. Chapter VIII contains an attempt to place the theory of resonance on a general basis, in a form which permits of application to specific problems. The point of view taken is mainly that of explaining how this phenomenon can be treated mathematically in certain of the cases of its presence in the solar system. It appears, however, to give a method of approach to the consideration of the question of the general stability of the orbits of the planets and thus leads to certain aspects of cosmogony. In Chapter IX, the theory is applied to the Trojan group of asteroids in a form which it is hoped will make the calculations of the orbits of these bodies easier than has hitherto been the case.

The appendix on Harmonic Analysis will be found to contain formulae for its application to the development of a given function ready for actual use. Most of these formulae have been tried out extensively and have been found to render the computations easy and accurate, especially when the number of such functions to be analysed is great.

We are indebted to Dr Dirk Brouwer and Mr R. I. Wolff for the errata given on p. xii and discovered after the sheets had been printed.

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1933

Errata

p. 26, line 5 from bottom, insert factor $\frac{1}{r^2}$ before $\frac{\partial R}{\partial \Gamma}$.

p. 28, line 6 from bottom, *for* $u^2 \frac{\partial R}{\partial u}$ *read* $\mu u^2 \frac{\partial R}{\partial u}$.

p. 28, Equation (4), *for* $\frac{1}{2}q$ *read* $\frac{1}{2q}$.

p. 28, Equation (5), *for* $\frac{2}{u^2}$ *read* $\frac{2q^2}{u^2}$.

p. 28, Equation (7), *for* $-\Gamma$ *read* $+\Gamma$.

p. 30, omit lines 12, 13.

p. 63, Equation (10), *for* ndt *read* $andt$.

p. 64, lines 12, 15, *for* $\left(\frac{1-e}{1+e}\right)^{\frac{1}{2}}$ *read* $\left(\frac{1+e}{1-e}\right)^{\frac{1}{2}}$.

p. 66, last line, *for* $4\pi^2\mu$ *read* $\mu/4\pi^2$.

p. 68, line 18, insert Σ_k before second formula.

p. 76, Equation (7), *for* $\left(\frac{e}{2}\right)^i$ *read* $\left(\frac{e}{2}\right)^j$.

p. 83, line 18, *for* a^2 *read* α^2.

p. 87, line 7, *for* $x = \exp\psi$ *read* $x = \exp\psi\sqrt{-1}$.

p. 87, Equation (2), delete the letter p.

p. 98, line 22, *for* $\sqrt{\frac{1-e}{1+e}}$ *read* $\sqrt{\frac{1+e}{1-e}}$.

p. 117, Equation (2), *for* m_i *read* m.

p. 118, line 5, *for* (5) *read* (6).

p. 129, line 6 from bottom, *for* (3) *read* (8).

p. 131, line 14, *for* $\mu(-2a_1)^{-\frac{3}{2}}$ *read* $\mu l(-2a_1)^{-\frac{3}{2}}$.

p. 131, delete line 15, replacing it by "Here the choice for S, slightly differing from (1), is".

p. 145, line 2, *for* $\frac{\partial S}{\partial t}$ *read* $-\frac{\partial S}{\partial t}$.

p. 157, line 4 from bottom, insert Σ before the last term.

p. 163, Equation (12), *for* -3Σ *read* $-3\Sigma j_1$.

p. 176, last line, *for* $D\mathrm{v}=1$ *read* $Dv=1$.

p. 224, line 15, *for* 8·8 (9) *read* 8·8 (10).

p. 230, line 5, *for* R *read* R_0.

p. 250, lines 5, 23, *for* Laplace *read* Lagrange.

p. 250, line 19, *for* configuration *read* condition.

p. 253, line 6 from bottom, *for* $\frac{2(m_0+m')}{r^3}$ *read* $\frac{2m_0}{r^3}$.

p. 255, line 11, the equations should read

$$\frac{aB}{A} = \frac{\pm 3\sqrt{3}\,m' - 8a^3n^2\iota\,(\frac{27}{4}m)^{\frac{1}{2}}}{-27a^3n^2m - 9m'} = \mp\frac{\sqrt{3}}{12} + \frac{\iota}{\sqrt{3m}}.$$

CHAPTER I

EQUATIONS OF MOTION

A. General Introduction

1·1. The methods considered in this volume for the investigation of the mutual actions of two or more bodies are based wholly on Newton's three laws of motion and on his law of gravitation. It is assumed that there exist fundamental frames of reference with respect to which the laws are exact and that the space in which the bodies move is Euclidean. The modern theory of relativity gives a different approach to the problem, but from the point of view taken here, which is chiefly that of deriving formulae for the comparison of gravitational theory with observation, the numerical difference resulting from the two methods of approach is very small, and can be exhibited as a correction to the results obtained through the Newtonian approach. These corrections, which are near the limit of observational accuracy at the present time, will not be considered here.

A further limitation is the treatment of the motions of the bodies as those of particles having masses equal to the actual masses. Here again, owing to the theorem that a sphere of matter, whose layers of equal density are concentric spheres, attracts an outside body as if it were a material particle, and also, owing to the fact that most of the bodies with which we have to deal are approximately spheres of this character or are sufficiently far away from the attracted body that they can be so treated, the differences are small. All other possible and actual forces, unless they obey the inverse square law and have constants which can be supposed to be included in the constants which we call the masses of the bodies, are neglected.

1·2. A general knowledge of the masses and relative distances of the various bodies from one another has to be assumed because the method of treatment to be recommended depends

on this knowledge. The two principal divisions are the planetary problem and the satellite or stellar problem. A third division may include the cometary problem and those cases of motions which cannot be included in either of the first two divisions.

In the planetary problem, one mass is very much greater than the combined masses of all the other bodies and dominates the motion of any one of them to such an extent that during a few revolutions the orbit of the latter is not greatly different from that which it would describe if the remaining masses did not exist. These approximate orbits moreover are ellipses with the principal mass in one focus and having minor axes which do not differ from their major axes by more than about ten per cent. of the latter. Further, the planes of these ellipses are inclined to one another at small angles—generally less than 20°. The distances of the bodies from one another may have any values whatever provided they do not fall below a certain limit. In general, the methods of this volume are developed for this case alone.

In the satellite or stellar problem the distance between two of the bodies must be small compared with the distance of either from the third; the two nearer bodies circulate round one another and their centre of mass circulates round the third body. The maintenance of this state of motion requires a limiting relation between the masses and distances of the bodies. There are also limitations concerning the shapes and positions of the orbits similar to those present in the planetary case. In the satellite problem, the mass of one of the two nearer bodies is small compared with that of the other, and the mass of the latter small compared with that of the third body. In the stellar problem, the masses are usually of the same order of magnitude. The methods adopted to obtain the motions in these two limited cases are not applicable to the cometary and other cases.

The methods developed below give expressions for the coordinates in terms of the time which serve to give the positions of the bodies over long intervals of time: the results are usually named the *general perturbations.* Practically all other cases have at present to be treated by the method of *special perturbations*

which consists in a completely numerical process of calculating the orbit over successive small arcs by 'mechanical integration.'

Practically all the problems which have hitherto been brought within the range of observation belong to one of these classes. There are numerous problems in the stellar universe in which the law of gravitation undoubtedly plays a dominant rôle. At the present time the deviations from rectilinear motion have not been directly observed, although such deviations have been inferred by statistical methods.

1·3. Since no method for the exact integration of the equations of motion of three or more bodies exists, devices for continued approximation to the actual motion are used. Sometimes these lead rapidly to the desired degree of accuracy; in other problems, the process may have to be repeated many times. In most cases, the first approximation is taken to be an ellipse and this is equivalent to a start with the two-body problem and a continuation with the calculation of the disturbing effects produced by the attractions of other bodies.

As far as possible these changes in position or 'perturbations,' as they are named, are expressed by sums of periodic terms which take the form of sines or cosines of angles directly proportional to the time, or to some variable which always changes in the same sense as the time. When, however, the number of such terms becomes too great for convenient numerical application, the terms of very long period are replaced by powers of the time or other adopted independent variable, and these powers are used in combination with the other periodic terms. In any case, the expressions which are obtained give reliable results for a limited interval of time only; all the periods with which we have to deal are determined from observation and therefore possess a limited degree of accuracy.

Expansions of functions in series, especially as sums of sines and cosines, thus play a large part in the work. The possibility of obtaining these expansions in such forms that numerical results may be deduced from them without too much labour,

usually depends on the presence of small constants or variables —'parameters'—in the coefficients of the periodic terms, and the expansions are partly made along powers of these parameters. Their orders of magnitude are important. In general, they consist of the eccentricities which rarely rise much above ·2, of the inclinations of the orbital planes to one another—usually less than 20°, and of the ratios of the distances.

For the majority of cases, this last ratio lies between ·4 and ·8, and the fact that we are compelled to expand in powers of so large a parameter is responsible for many of the difficulties of the problems.

1·4. The most fundamental difficulty, however, is caused by approximate or exact 'resonance.' This term refers to those cases in which two or more of the periods, which enter into the expansions for the coordinates, are nearly or exactly in the ratio of two small integers. In the approximate case, large amplitudes of certain of the periodic terms, and slow numerical convergence to the needed degree of accuracy, are characteristic effects. In the cases of exact resonance the form of the solution has to be changed.

In either event, the terms which cause the chief trouble are those with periods which are long in comparison with the period of revolution of the body round the central mass. Such periodic terms may have small coefficients in the equations of motion, but the integration of the equations produces small divisors which give large coefficients in the coordinates. These small divisors demand that the terms affected be carried to a much higher degree of accuracy than the remaining terms, and as there exists no short method for securing this accuracy, the amount of calculation needed in any given planetary problem depends mainly on the few, perhaps one or two only, terms of long period which are sensible in the observations. The existence of such terms in every planetary problem has to be kept in mind while divising methods and in carrying them out. The method finally chosen should depend mainly, not on the ease

with which the first approximation may be obtained, but on the work required for the final approximation.

1·5. *Astronomical measurements.* The only measurements of the position of a celestial object which have a precision comparable with gravitational theory are those of angles on the celestial sphere. The ultimate planes of reference from which these angles are measured along great circles are defined by the average positions of the stars (which have observable motions relative to one another). For the theory, an origin is needed, and this is ultimately the centre of mass of the system. It is assumed that these definitions will give a Newtonian frame because the stars are so far away that they affect neither the motion of the centre of mass nor the relative motions of the bodies within the system to an observable extent.

Time in this frame is measured by the interval between the instants when a plane fixed in the earth and passing through its axis of rotation passes through a mark in the sky supposed to be fixed relatively to the stars.

Owing to the rapid motion of the earth about its axis the observer finds it convenient to give his measurements with respect to the plane of the earth's equator, and to a point on the equator defined by its intersection with the ecliptic—the plane of the earth's orbit round the sun. Both these planes are in motion but their motions and positions relative to the ultimate stellar frame are supposed to be known. This frame is inconvenient for working out gravitational theories on account of its large inclination to the planes of motion of most of the bodies of the solar system. For this purpose the ecliptic and the point on it defined above are used. The motions of these are nearly uniform and are easily taken into account.

In the observer's frame, the angular coordinates are the *declination* measured along a great circle from the object to the equator and perpendicular to the latter, and the *right ascension,* the angle between this great circle and that perpendicular to the equator and passing through its intersection with the

ecliptic. In the computer's frame, the angular coordinates are the celestial *latitude* and *longitude* similarly measured with respect to the ecliptic. As this latter frame is moving the ultimate reference is to its position at some given date. Thus to transform the results obtained from the gravitational theory for the use of the observer, geometrical relations must be computed, and there is, in addition, a kinematical relation due to the motion of the observer's frame.

It is assumed that a complete gravitational theory of the motion of any body in the solar system referred to a frame whose motions are fully known, should give the position of the body at any time when the constants of the motion have been determined. Differences between the calculated and observed positions of the body may be due to defects in the theory, unknown motions of the frame of reference, errors in the determination of the constants, or errors of observation. The analysis of these differences in order to discover their source is a problem involving many difficulties. In many cases two or more interpretations are possible and these can only be separated by the use of more observations. An outstanding difference, for example, between the observed and calculated values of the motion of the perihelion of Mercury was variously attributed to the gravitational attraction of a ring of matter supposed to surround the sun, to a motion of the frame of reference, to defects in the gravitational theory, until the theory of relativity furnished an explanation. A marked deviation of the moon from its gravitational theory has received an explanation as a variation in the rate of rotation of the earth about its axis, through detection of similar deviations from the gravitational theories of the observed position of the sun, the satellites of Jupiter, and the planet Mercury.

1·6. Observations of bodies in the solar system are usually of two classes. Those made with the transit telescope give the instant of passage of the body across the meridian of the observer and the angle between its observed direction and that of the earth's axis, the time being given by a clock which is constantly compared with the transits of stars. Differential observations, often made by photography, give the position of the body at any time with respect to stars in its neighbourhood, the places of these stars referred to the frame being known. Under good conditions, either class of observation should give the position with a probable error less than 1″—such a standard at least is aimed at in the gravitational theories of the principal bodies in the solar system.

While direct observations of distances cannot in general be made accurately, their theoretical determination is necessary because we cannot

eliminate them from the equations of motion without introducing complications greater than those which the equations already possess. They are also needed because the planetary theories use the sun as an origin; the transformation from the observer's position on the earth to an origin in the sun requires a knowledge of the variations of these distances.

Observations of masses or of relative masses are not made: a mass is only known to us by its gravitational effects. In the solar system, the orders of magnitude are such that the mass of a body has little effect on its own motion; the operation of Kepler's third law, (3·6), eliminates from the angular coordinates the greater part of the mass effect. When dealing with the effect of one planet on another, we need to know the ratio of the mass of the disturbing planet to that of the sun. If the planet has a satellite whose motion can be calculated, the mass of the planet can be found with sufficient accuracy to calculate its effect on any other planet. When it has no satellite, its mass can be found only by comparing its calculated disturbing effects with those furnished by observation. There is of course a correlation between the degree of accuracy required to calculate the perturbations and that with which the perturbations can be observed. Thus the masses of all the major planets except those of Venus and Mercury are fairly well determined. The masses of the minor planets can be obtained only from observation of the light they send with estimates as to their albedo and density, since they are too small to exert any observable attractive effect on any other body.

1·7. *The Newtonian law of gravitation.* This law states that the attractive force between two particles of masses m, m' at a distance r apart is along the line joining them and of magnitude Cmm'/r^2, where C is a constant—the 'gravitation constant'—whose value depends only on the units adopted. It is convenient, in order to avoid the continual presence of C in the mathematical operations, to so choose the units that $C=1$. Since m/r^2 has then the apparent dimensions of an acceleration, it follows that a mass with these units has the apparent dimensions of the cube of a length and the inverse square of a time. The adoption of this unit—called the 'astronomical unit of mass'—is closely associated with Kepler's third law. This law, with a slight modification (3·6) to bring it into accord with the law of gravitation, states that if $2\pi/n$ be the period of revolution of two bodies about one another and if a be their mean distance apart,

$$n^2a^3 = C \times \text{sum of masses}.$$

In numerical work we need to deal with ratios only. If a relation of this kind is used in the equations of motion, the latter will be freed from the apparent inconsistency of possessing terms having different physical dimensions, and will consist of ratios of masses, distances and times only.

The units of time and distance to be finally chosen, depend on the problem under investigation: a choice is not usually needed until comparison with observation is to be made. For this purpose, the mean solar second, day or year are used as units of time, and the mean radius of the earth or the mean distance of the earth from the sun as units of distance.

B. Planetary and Satellite or Stellar Types of the Equations of Motion

1·8. *The Equations of motion with rectangular coordinates* A *force-function* for the motion of any particle, when it exists, is defined as a function whose directional derivatives with respect to the coordinates of the particle give the components of the forces.

For two particles with masses m, m' and at a distance r apart, the gravitational force-function is mm'/r. For $n+1$ bodies with masses m_i and mutual distances r_{ij}, it is

$$V = \Sigma_{ij} \frac{m_i m_j}{r_{ij}}, \qquad i, j = 0, 1, \ldots, n; \; i \neq j,$$

the summation including each combination of i, j once only.

This function is independent of the directions in the frame of reference which may be used. It may or may not depend on the origin chosen.

In treatises on general mechanics, the *potential* is defined as a function which has the property that its directional derivatives give the *reversed* components of all the forces which act on the system. We shall have occasion later (e.g. in 1·9) to construct force-functions which are not potentials with reversed signs, since the force-function for the motion of one particle is not the same as those for the other particles of the system.

Let ξ_i, η_i, ζ_i be the coordinates of m_i. Then, with the definition of V given above and under the limitations stated in 1·1, the equations of motion are

$$m_i \frac{d^2\xi_i}{dt^2} = \frac{\partial V}{\partial \xi_i}, \quad m_i \frac{d^2\eta_i}{dt^2} = \frac{\partial V}{\partial \eta_i}, \quad m_i \frac{d^2\zeta_i}{dt^2} = \frac{\partial V}{\partial \zeta_i}. \quad \ldots(1)$$

Only the relative motions of the bodies are needed and, in finding them, the equations are usually given two principal forms, depending on the origins chosen.

1·9. *Planetary form of the equations.* In this form, one body m_0 is chosen as the origin of coordinates, and the motions of the remaining bodies relative to m_0 are to be determined.

Let

$$x_k = \xi_k - \xi_0, \quad y_k = \eta_k - \eta_0, \quad z_k = \zeta_k - \zeta_0, \qquad k = 1, 2, \ldots n,$$

$$r_k^2 = r_{0k}^2 = x_k^2 + y_k^2 + z_k^2.$$

Then $$r_{jk}^2 = (x_k - x_j)^2 + (y_k - y_j)^2 + (z_k - z_j)^2.$$

These definitions, together with the equations of motion, give

$$\frac{d^2x_k}{dt^2} = \frac{1}{m_k}\frac{\partial V}{\partial \xi_k} - \frac{1}{m_0}\frac{\partial V}{\partial \xi_0}$$

$$= \frac{\partial}{\partial x_k}\left(\frac{m_0}{r_k} + \Sigma_j \frac{m_j}{r_{jk}}\right) - \frac{m_k x_k}{r_k^3} - \Sigma_j \frac{m_j x_j}{r_j^3},$$

where $j \neq k$. The penultimate term, being equal to the derivative of m_k/r_k with respect to x_k, can evidently be combined with that of m_0/r_k. The last term can be written

$$-\frac{\partial}{\partial x_k}\Sigma_j m_j \frac{x_j x_k + y_j y_k + z_j z_k}{r_j^3}, \qquad j \neq k,$$

in which form it will serve for all three coordinates.

Hence, if we put

$$R_k = \Sigma_j m_j\left(\frac{1}{r_{jk}} - \frac{x_j x_k + y_j y_k + z_j z_k}{r_j^3}\right), \quad \ldots\ldots\ldots(1)$$

the equations of motion for m_k relative to m_0 become

$$\frac{d^2x_k}{dt^2} = \frac{\partial}{\partial x_k}\left(\frac{m_0 + m_k}{r_k} + R_k\right), \quad \ldots\ldots\ldots\ldots\ldots(2)$$

with similar equations for y_k, z_k.

so that the new coordinates still depend on the differences of the original coordinates.

Now when V is expressed in terms of the coordinates ξ_i, η_i, ζ_i, we have, as a result of changes in the ξ_i only,

$$dV = \frac{\partial V}{\partial \xi_0} d\xi_0 + \frac{\partial V}{\partial \xi_1} d\xi_1 + \frac{\partial V}{\partial \xi_2} d\xi_2.$$

But since V contains the ξ_i only through their differences.

$$\frac{\partial V}{\partial \xi_0} + \frac{\partial V}{\partial \xi_1} + \frac{\partial V}{\partial \xi_2} = 0. \quad \text{.....................(3)}$$

Whence, combining this with the previous equation, we have

$$dV = \frac{\partial V}{\partial \xi_1}(d\xi_1 - d\xi_0) + \frac{\partial V}{\partial \xi_2}(d\xi_2 - d\xi_0).$$

But when V is to be expressed in terms of x, x', we have

$$dV = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial x'} dx'.$$

If we substitute the values for dx, dx' in terms of $d\xi_1 - d\xi_0$, $d\xi_2 - d\xi_0$ and equate the coefficients of the latter in the two expressions for dV, we obtain

$$\frac{\partial V}{\partial \xi_1} = -\frac{\partial V}{\partial x} - \frac{m_1}{m_1 + m_2}\frac{\partial V}{\partial x'}, \quad \frac{\partial V}{\partial \xi_2} = +\frac{\partial V}{\partial x} - \frac{m_2}{m_1 + m_2}\frac{\partial V}{\partial x'}. \quad \text{......(4)}$$

The transformed equations of motion for x, x' become, with the help of these results,

$$\frac{d^2x}{dt^2} = \frac{1}{m_2}\frac{\partial V}{\partial \xi_2} - \frac{1}{m_1}\frac{\partial V}{\partial \xi_1} = \frac{m_1 + m_2}{m_1 m_2}\frac{\partial V}{\partial x}, \quad \text{.........(5)}$$

$$\begin{aligned}\frac{d^2x'}{dt^2} &= \frac{1}{m_0}\frac{\partial V}{\partial \xi_0} - \frac{1}{m_1 + m_2}\left(\frac{\partial V}{\partial \xi_1} + \frac{\partial V}{\partial \xi_2}\right) \\ &= -\left(\frac{1}{m_0} + \frac{1}{m_1 + m_2}\right)\left(\frac{\partial V}{\partial \xi_1} + \frac{\partial V}{\partial \xi_2}\right), \text{ by (3)}, \\ &= \frac{m_0 + m_1 + m_2}{m_0(m_1 + m_2)}\frac{\partial V}{\partial x'}, \text{ by (4)}.\end{aligned}$$

For the purposes of calculation, we insert the value of V, namely,

$$V = \frac{m_0 m_1}{r_{01}} + \frac{m_0 m_2}{r_{02}} + \frac{m_1 m_2}{r_{12}},$$

with $$r_{12}^2 = x^2 + y^2 + z^2 = r^2,$$

$$r_{01}^2 = \left(x' + \frac{m_2}{m_1 + m_2} x\right)^2 + \dots + \dots,$$

$$r_{02}^2 = \left(x' - \frac{m_1}{m_1 + m_2} x\right)^2 + \dots + \dots,$$

giving

$$\frac{d^2 x}{dt^2} = \frac{\partial}{\partial x}\left\{\frac{m_1 + m_2}{r} + \frac{m_0(m_1 + m_2)}{m_1 m_2}\left(\frac{m_1}{r_{01}} + \frac{m_2}{r_{02}}\right)\right\}, \quad \dots\dots(6)$$

$$\frac{d^2 x'}{dt^2} = \frac{\partial}{\partial x'}\left\{\frac{m_0 + m_1 + m_2}{m_1 + m_2}\left(\frac{m_1}{r_{01}} + \frac{m_2}{r_{02}}\right)\right\}. \quad \dots\dots\dots\dots\dots\dots(7)$$

For theoretical investigations, the equations are exhibited in the Newtonian form by putting

$$\mu = \frac{m_1 + m_2}{m_1 m_2}, \quad \mu' = \frac{m_0 + m_1 + m_2}{m_0(m_1 + m_2)}, \quad V_1 = \mu\mu' V, \quad \dots\dots(8)$$

when they become

$$\mu \frac{d^2 x}{dt^2} = \frac{\partial V_1}{\partial x}, \quad \mu' \frac{d^2 x'}{dt^2} = \frac{\partial V_1}{\partial x'}, \quad \dots\dots\dots\dots(9)$$

with similar equations for y, z, y', z'.

This form cannot be used, however, if m_1 or m_2 is zero. But the equations then revert to the planetary form, with the motion of m_0 relative to m_1 elliptic if $m_2 = 0$, so that the only motion which has to be considered is that of m_2 relative to m_1.

1·13. The equations of motion in the form 1·12 (6), (7) will not be needed in the developments of this volume since they are, in general, useful only when the distance between m_1, m_2 is small compared with those between m_0, m_1 and m_0, m_2, that is when r/r' is small. The initial development of V will, however, be shown, in order to exhibit the contrast with the developments used in the planetary problem.

Put $$xx' + yy' + zz' = rr' \cos S, \quad x'^2 + y'^2 + z'^2 = r'^2,$$

so that S is the angle between r, r'. Then

$$r_{01}^2 = r'^2 + \frac{2m_2}{m_1+m_2} rr' \cos S + \left(\frac{m_2}{m_1+m_2}\right)^2 r^2,$$

$$r_{02}^2 = r'^2 - \frac{2m_1}{m_1+m_2} rr' \cos S + \left(\frac{m_1}{m_1+m_2}\right)^2 r^2.$$

Hence, if P_i be the zonal harmonic of degree i with argument S, we have

$$\frac{1}{r_{01}} = \frac{1}{r'}\left\{1 + \sum_{i=1}^{\infty}\left(-\frac{m_2}{m_1+m_2}\frac{r}{r'}\right)^i P_i\right\},$$

$$\frac{1}{r_{02}} = \frac{1}{r'}\left\{1 + \sum_{i=1}^{\infty}\left(\frac{m_1}{m_1+m_2}\frac{r}{r'}\right)^i P_i\right\},$$

and thence

$$V = \frac{m_1 m_2}{r} + \frac{m_0(m_1+m_2)}{r'} + \frac{m_0}{r'}\sum_{i=2}^{\infty}\frac{m_2 m_1^i + m_1(-m_2)^i}{(m_1+m_2)^i}\left(\frac{r}{r'}\right)^i P_i,$$

the terms for $i=1$ having disappeared.

The force-function 1·12 (6) then becomes

$$\frac{m_1+m_2}{r}\left\{1 + \frac{m_0}{m_1+m_2}\frac{r^3}{r'^3}\sum_{i=1}^{\infty}\frac{m_1^i - (-m_2)^i}{(m_1+m_2)^i}\left(\frac{r}{r'}\right)^{i-1} P_{i+1}\right\}, \quad \text{......(1)}$$

the second term of V being useless. The function 1·12 (7) becomes

$$\frac{m_0+m_1+m_2}{r'}\left\{1 + \frac{m_1 m_2}{(m_1+m_2)^2}\frac{r^2}{r'^2}\sum_{i=1}^{\infty}\frac{m_1^i - (-m_2)^i}{(m_1+m_2)^i}\left(\frac{r}{r'}\right)^{i-1} P_{i+1}\right\}, \quad \text{......(2)}$$

the first term in this case being useless.

In each case, if the function be confined to its first term we obtain elliptic motion (Chap. III), the remaining portion being that which produces the disturbing effect. If n, a be the mean angular velocity and mean distance in the former, and n', a' those in the latter, we have by Kepler's third law, (3·6),

$$m_1 + m_2 = n^2 a^3, \quad m_0 + m_1 + m_2 = n'^2 a'^3.$$

By putting $i=1$, we see that the significant factors of the disturbing effects in (1), (2) are

$$\frac{m_0}{m_1+m_2}\frac{a^3}{a'^3} = \frac{m_0}{m_0+m_1+m_2}\frac{n'^2}{n^2}, \quad \frac{m_1 m_2}{(m_1+m_2)^2}\frac{a^2}{a'^2}.$$

In the satellite problem, m_0 is much greater than m_1 and m_1 than m_2, and a/a', n'/n are small, so that both these factors are small. If m_0, m_1, m_2 refer to the sun, earth and moon, respectively, these ratios have the magnitudes ·007, $1{\cdot}3 \times 10^{-7}$.

In those stellar problems which have up to the present shown observational evidence of perturbing effects, m_0, m_1, m_2 are of the same order of magnitude, but a/a', n'/n are small—of the order ·2 or less. The disturbing effects are chiefly shown in the motions of the apses and nodes. See P. Slavenas, "The Stellar Case of the Problem of Three Bodies*."

* *Trans. of Yale Obs.* vol. 6, pt. 3.

C. Frames of Reference

1·14. *Choice of variables.* In the preceding sections, the equations of motion have been referred to certain origins in a Newtonian frame with fixed directions for axes. Experience has shown, however, that neither rectangular coordinates nor fixed axes are convenient for finding the position of the body at any time, and that the calculations may be much abbreviated by suitable choices of coordinates. Analytically, the deduction of these sets of equations may be regarded as nothing more than a change from one set of variables to another. But since the choice of a set of variables always depends on a knowledge of the general characteristics of the motion, it is often useful to give a geometrical or dynamical interpretation to the variables chosen.

1·15. For the development of the planetary theory, the *osculating plane* as a principal plane of reference possesses certain advantages over all other planes of reference. It is defined as a plane passing through the sun and the tangent to the orbit of the planet. The plane is in motion but it is found in most cases that its motions are either small or slow; that is, its deviations from a mean position are either small or require long periods of time to become large. This fact can be so used in the analytical work as to abbreviate the calculations.

A second and even more important property of this plane is due to the small effect its motion has on the motion of the planet within the plane. In many cases this secondary effect can be altogether neglected, so that the motion within the plane can be treated as though the latter always occupied its mean position. These remarks refer equally to both disturbing and disturbed planet, that is, to the planet whose motion is supposed known and to that which we are finding. The effect of the motion of the plane of the former on the latter is usually negligible or can be accounted for quite simply.

All the methods developed in this volume use the osculating plane as a plane of reference.

1·16. The choices of coordinates within the plane of reference may be placed in two categories. In the first of these, the distance of the planet from the sun or some function of this distance is used as one coordinate, the second coordinate being the elongation of this radius reckoned from some fixed or moving line with the time as the third variable in the equations of motion. The rôles played by the second and third of these variables may be interchanged.

Two ways of measuring the elongation are used below. One is the usual method of using a single symbol to denote the sum of two angles measured in different planes: that between the radius and the line of intersection of the osculating plane with a fixed plane of reference, and that between this latter line and a line fixed* in the plane of reference. This symbol is usually called the longitude in the orbit or, briefly, the longitude. The second is the measurement of the elongation from a line in the osculating plane, this line being so defined that its resultant velocity is always perpendicular to the osculating plane. This second method has the advantage of eliminating the motion of the osculating plane from the kinetic reactions within the plane. With the use of these methods it is convenient to introduce an auxiliary variable which substantially is the angular momentum or a function of it.

1·17. The second category consists of the use of certain variables associated with the *osculating ellipse.* This ellipse is defined as the orbit which the body would follow if, at any instant, all disturbing forces were annihilated and the body continued its motion under the sole attraction of the central mass. The definition requires that the velocity in the orbit and in the osculating ellipse shall be the same in magnitude and direction and therefore that its plane shall be the osculating plane at the point. The variables used are those which define the size, shape and position of the ellipse, or certain functions of them which may or may not contain the time. These functions are called the *elements* of the ellipse.

The elements which are simplest for descriptive purposes are

* The word 'fixed' is used in the Newtonian sense.

the major axis, the eccentricity, the longitude of the axis and the time of passage through the nearer apse or position when the distance from the focus occupied by the central mass is least; the period of revolution is connected with the major axis by Kepler's third law which involves the sum of the masses of the two bodies. This sum may be unknown but, as it remains constant, the relation between the variations of the major axis and the period is always the same. Various combinations of these elements and of the two elements which define the position of the osculating plane are also used as elements: as we go from point to point of the actual orbit these elements will change. According to the definition, the changes will depend on the existence of attractions other than that of the central mass, and the term 'Variation of the Elements' refers to these changes. It will be seen below that the methods used to determine them are similar to that unfortunately named 'the Variation of Arbitrary Constants' in the theory of differential equations.

1·18. This geometrical description of the elliptic frame, while useful for descriptive purposes, conceals the analytical meaning which is essential for a clear understanding of the processes involved. *Analytically, the elements are nothing else than a new set of variables allied to the coordinates by a definite set of relations which remain unchanged.* Thus the process of forming the differential equations satisfied by the elements is precisely that of changing from one set of variables to another.

The description of the process is complicated by the fact that the *three* old variables (the coordinates) are replaced by *six* new variables; consequently, three relations between the latter are at our disposal. If the coordinates be denoted by x_i and the new variables by a_j, and if the relations between them be

$$x_i = f_i(a_1, a_2, \dots, a_6, t), \qquad i = 1, 2, 3,$$

then the three additional relations are almost invariably chosen to be so defined that they satisfy the equations

$$\frac{dx_i}{dt} = \frac{\partial f_i}{\partial t}.$$

As a result of this definition, we have

$$\Sigma_j \frac{\partial f_i}{\partial \alpha_j} \frac{d\alpha_j}{dt} = 0. \qquad \text{.......................(1)}$$

It follows that x_i, dx_i/dt are replaced by f_i, $\partial f_i/\partial t$ in the equations of motion and that d^2x_i/dt^2 is replaced by

$$\Sigma_j \frac{\partial^2 f_i}{\partial \alpha_j \partial t} \frac{d\alpha_j}{dt} + \frac{\partial^2 f_i}{\partial t^2}. \qquad \text{.....................(2)}$$

This last process gives three equations and these with (1) furnish the six equations necessary to find the α_j. The forms of the functions f_i, and consequently those of their partial derivatives, remain unchanged and are given by algebraic and trigonometrical formulae developed in Chap. III.

The analytical point of view just given is that which is chiefly needed in the development of the equations of motion. This view is often obscured by the methods used to obtain the functions f_i. These methods require the solution of the equations for elliptic motion and in this solution the α_i appear as the arbitrary constants: in the general problem they become the variables. The fact is that the solution of the equations for elliptic motion is merely a convenient device for finding the functions f_i which connect the old and new variables.

The fact that the differential equations satisfied by the new variables are all of the first order, together with another property to be developed in Chap. V, namely, that the variables can be so chosen that the equations have the canonical form, is largely responsible for the use that has been made of them in theoretical investigations. Their practical value lies in the ease with which the equations may be integrated and in the simplicity of the geometrical interpretations which may be given to some of the results.

1·19. Certain methods like those of Hansen and Gyldèn, not developed in this volume, as well as that given in Chap. VII, possess to some extent the characteristics of both categories. No sharp division is possible or necessary, the sole test being that of convenience for the problem under consideration. Whenever a new variable is introduced, it can generally be related to some property of the ellipse, but this relation is not usually helpful except in so far as it may have led to the choice of the variable.

1·20. The frames of reference should also be regarded as four-dimensional in the sense that the time as well as the space coordinates should enter into consideration in making choices of new variables. The observer's demand is for expressions giving the space coordinates in terms of the time, but the analyst is free to regard any three of them as a function of the fourth and to solve the problem according to his choice. If the time be not used as the fourth coordinate, that is, as the independent variable, a final transformation is usually, though not necessarily, made to obtain the space coordinates in terms of the time. However, practical demands limit the nature of the independent variable. The linear coordinates are sums of periodic functions of the time, that is of a variable which is unlimited in magnitude and whose changes are always in the same sense. Any other independent variable which is chosen should have the same property: it may be angular or even areal provided the angles or areas are always changing in the same sense. Otherwise troublesome infinities are apt to be introduced.

1·21. The choice of a method for the solution of any particular problem depends on a number of factors which should receive consideration.

As between the two principal categories described above (1·16 and 1·17), the elliptic frame requires the calculation of the expressions for six variables as against the three coordinates which are alone needed by the observer. On the other hand, the solution of the differential equations is much more simple for the elliptic frame than those for the coordinates. For this reason, certain sets of equations belonging to the first category have been so developed that their solution is as simple as those for the elliptic elements.

The elliptic frame as actually used requires a literal development of the disturbing forces in terms of the variables: when high accuracy is needed this development may entail very great labour. On the other hand, in particular portions of the problem, for example, in the calculation of the secular terms, those of very long period and resonance terms, it appears to give the needed results more easily than any other method which has had extensive trial.

For theoretical researches, and for the discovery of qualitative properties of the motion, the elliptic frame has in general been more fruitful than most of the other forms of the equations of motion. This statement refers to motions of the general character of those which have been observed

rather than to those which are mathematically possible, and to work which has been done in the past rather than to what may be accomplished in the future. In this connection, it should be remembered that a quantitative solution for a particular set of problems is often more easily obtained by a procedure different from that which is used to deduce a qualitative result.

1·22. In making a choice for the solution of a particular problem from the various methods which have been proposed or developed, there are several factors which should receive consideration.

1. The question as to whether a literal or numerical development is to be made, that is, a development available for several cases or one which is applicable to the motion of a single body only. The choice depends, not only on the number of cases to which the solution can be applied, but on the degree of accuracy with which the initial conditions, that is, the arbitrary constants, are known. In cases where the deviations from elliptic motion are large, the literal method may involve such extensive computations that it becomes practically impossible, even if the infinite series used were sufficiently convergent to give the quantitative results needed. Sometimes a partly literal and partly numerical method can be adopted with but little extra labour. In all numerical methods, provision must be made for changes in the arbitrary constants which future observations may furnish.

Some details with reference to the problems of the solar system will make these statements more concrete. For the eight major planets the elements are known with considerable accuracy so that corrections to them need scarcely be considered at the present time as a factor in the choice of a method. It is impracticable to use the literal values of the ratios of their mean distances from one another owing to the numerical magnitudes of these quantities: numerical values must be adopted for these from the outset and these involve numerical values for the periods of revolution round the sun. Little is gained by the use of literal values for their eccentricities and inclinations, and much labour is saved by using numerical values for the constant parts of the angular elements. Thus, completely numerical theories are indicated for the major planets. For the moon, the ratio of the periods of the moon and sun is the parameter along which convergence is least rapid and there is little doubt that its numerical value should be used from the outset. Literal values for all the other elements can be used with but little additional work.

For the minor planets, numerical values of the ratios of the mean distances are again a necessity, but since there are groups of them in which this ratio is nearly the same it is useful to devise methods in which this ratio has a given numerical value while the other elements are left

arbitrary. For most of the satellites other than the moon, complete numerical theories are indicated. This becomes a practical necessity in the cases of the outer satellites of Jupiter where the eccentricities and inclinations have large values, although nothing is gained by using the numerical values of the constant parts of the angles.

In general, planetary problems should be separated from satellite problems. In the former convergence is slow along powers of the ratios of the mean distances, but rapid along powers of the ratio of the mass of the disturbing body to that of the primary; in the latter the case is reversed. For the planetary problems the amount of calculation needed for the terms dependent on the second and higher powers of the mass ratio is nearly always small compared with that needed for the first power, except, perhaps, in the case of the mutual perturbations of Jupiter and Saturn. The same is true in the asteroid problems, except in the difficult resonance cases, on account of the lower degree of accuracy at present demanded.

2. Consideration should be given to the amount of routine computing available. In some methods much of the work can be arranged so as to be done by routine computers, in others this is not the case.

3. The liability to errors of computation and the extent to which tests may be applied, play some part. It is rare that an extensive theory is tested throughout by others than the author, and safeguards against mistakes should be provided as far as possible.

4. Possibilities for an extension of the work as new needs arise.

5. An examination of existing developments in order to discover the numerical magnitudes which will be involved in the work.

6. The degree of numerical accuracy aimed at.

7. The extent to which use can be made of existing numerical or literal developments and in particular of those of the disturbing function.

8. The extent to which any peculiarity of the motion may dominate the whole work. In most cases this peculiarity is that of approximate resonance between two periods, as for example in the great inequality in the motions of Saturn and Jupiter, the principal librations in the Trojan group of asteroids, and so on.

D. Various forms of the Equations of Motion depending on the use of Polar Coordinates in the Osculating Plane

1·23. *Polar coordinates with the time as independent variable.*

The osculating plane is one containing the origin and the tangent to the orbit of the disturbed body. It is defined by the angle i which it makes with a fixed plane and the angle θ which its line of intersection with that plane—the line of nodes—makes with a fixed line in the same plane. When i is less than 90°, θ is measured in the same sense as the actual motion; thus θ refers to that node at which the body is ascending from below to above the fixed plane. In Fig. 1, let O, ☊, P be the points where the fixed line, the line of nodes and the radius vector r cut the unit sphere with centre at the origin. Let the angle ☊P be denoted by $v - \theta$.

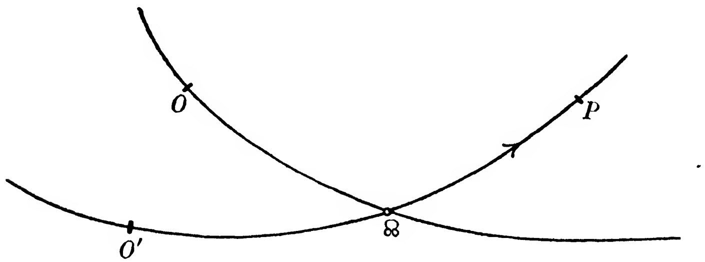

Fig. 1.

The angular velocity of ☊ along the fixed plane can be resolved into the components

$$\frac{d\theta}{dt}\cos i, \quad \frac{d\theta}{dt}\sin i,$$

within and perpendicular to the osculating plane. The latter contributes a component $-(d\theta/dt)\sin i\cos(v-\theta)$ to the motion of P perpendicular to the osculating plane.

The change of inclination contributes a velocity

$$(di/dt)\sin(v-\theta)$$

to the motion of P perpendicular to the same plane. The definition of this plane therefore gives

$$\frac{di}{dt}\sin(v-\theta)-\frac{d\theta}{dt}\sin i\cos(v-\theta)=0. \quad \text{.........(1)}$$

The velocity of P is compounded of its velocity relative to ☊ and the velocity of ☊. It is therefore

$$\frac{d}{dt}(v-\theta)+\frac{d\theta}{dt}\cos i=\frac{dv}{dt}-\Gamma\frac{d\theta}{dt}, \qquad \Gamma\equiv 1-\cos i.$$

Hence the square of the velocity of the planet is

$$2T=\left(\frac{dr}{dt}\right)^2+r^2\left(\frac{dv}{dt}-\Gamma\frac{d\theta}{dt}\right)^2.$$

A geometrical interpretation can be given to a variable v defined by

$$d\mathrm{v}=dv-\Gamma d\theta.$$

This definition makes $d\mathrm{v}/dt$ the angular velocity of the radius vector in the osculating plane. We can therefore regard v as an angle reckoned from a departure point O' in that plane which is such that, as the plane moves, the locus of O' is perpendicular to the trace of the osculating plane on the unit sphere.

The function T contains four variables. It may be regarded as the kinetic energy of a system with four degrees of freedom if we suppose the osculating plane to be material and if we add a term depending on its mass and motion. Let F be the force-function of this dynamical system. We can then apply Lagrange's equations to it, with the variables r, v, θ, Γ, and, after forming them, put the mass of the osculating plane equal to zero*. If, in the resulting equations, we put

$$\frac{dv}{dt}-\Gamma\frac{d\theta}{dt}=\frac{d\mathrm{v}}{dt}, \quad \text{.......................(2)}$$

they can be written

$$\frac{d^2r}{dt^2}-r\left(\frac{d\mathrm{v}}{dt}\right)^2=\frac{\partial F}{\partial r}, \quad \frac{d}{dt}\left(r^2\frac{d\mathrm{v}}{dt}\right)=\frac{\partial F}{\partial v}, \quad \text{...(3), (4)}$$

$$\frac{d}{dt}\left(\Gamma r^2\frac{d\mathrm{v}}{dt}\right)=-\frac{\partial F}{\partial \theta}, \qquad r^2\frac{d\mathrm{v}}{dt}\frac{d\theta}{dt}=\frac{\partial F}{\partial \Gamma}. \quad \text{...(5), (6)}$$

* The relation (1) expresses the fact that the material osculating plane is not acted on by any forces which do work.

In these equations* we substitute for F the force-function for the particular problem under consideration.

If x, y, z be the rectangular coordinates with the fixed plane as that of x, y and O as the trace of the x-axis, the positive part of the y-axis being 90° from O reckoned in the same sense as θ, and that of the z-axis being above the plane, we have

$$\left.\begin{aligned} x &= r\cos(v-\theta)\cos\theta - r\sin(v-\theta)\sin\theta\cos i,\\ y &= r\cos(v-\theta)\sin\theta + r\sin(v-\theta)\cos\theta\cos i,\\ z &= r\sin(v-\theta)\sin i = r\sin L, \end{aligned}\right\}\quad\ldots(7)$$

which show that F is expressible in terms of r, v, θ, Γ. The definition of L shows that it is the angle between r and its projection on the fixed plane or the latitude of the body above this plane.

1·24. *Canonical form of the equations.* If we put

$$G = r^2\frac{dv}{dt},\qquad H_1 = -\Gamma G,\ \ldots\ldots\ldots\ldots(1),\ (2)$$

so that

$$2T = \left(\frac{dr}{dt}\right)^2 + \frac{G^2}{r^2},\qquad \frac{dr}{dt} = \frac{\partial T}{\partial \dot r},\qquad \frac{G}{r^3} = -\frac{\partial T}{\partial r},$$

we can write the equations in the form

$$\left.\begin{aligned} \frac{d\dot r}{dt} &= \frac{\partial}{\partial r}(F-T), & \frac{dr}{dt} &= -\frac{\partial}{\partial \dot r}(F-T),\\ \frac{dG}{dt} &= \frac{\partial}{\partial v}(F-T), & \frac{dv}{dt} &= -\frac{\partial}{\partial G}(F-T),\\ \frac{dH_1}{dt} &= \frac{\partial}{\partial\theta}(F-T), & G\frac{d\theta}{dt} &= \frac{\partial}{\partial\Gamma}(F-T), \end{aligned}\right\}\quad\ldots\ldots(3)$$

or, more compactly,

$$d\dot r\,.\,\delta r - \delta\dot r\,.\,dr + dG\,.\,\delta v - \delta G\,.\,dv$$
$$+\,dH_1\,.\,\delta\theta + G d\theta\,.\,\delta\Gamma = \delta\,(F-T)\,.\,dt.$$

* For other derivations of these equations, see E. W. Brown, "Theory of the Trojan Group of Asteroids," *Trans. of Yale Obs.* vol. 3 (1923), p. 9; C. A. Shook, "An Extension of Lagrange's Equations," *Bull. of the Amer. Math. Soc.* vol. 38 (1932), p. 135.

If, in the latter, we replace dv by $dv - \Gamma d\theta$ and $Gd\theta \,.\, \delta\Gamma$ by $d\theta\,(-\delta H_1 - \Gamma\delta G)$, obtained by submitting (2) to a variation δ, the equations take the canonical form

$$d\dot{r} \,.\, \delta r - \delta\dot{r} \,.\, dr + dG \,.\, \delta v - \delta G \,.\, dv + dH_1 \,.\, \delta\theta - \delta H_1 \,.\, d\theta = \delta\,(F - T) \,.\, dt. \quad \text{...(4)}$$

We have here implicitly supposed that Γ has been replaced by $-H_1/G$ in F; T is a function of $\dot{r}$, G, r only.

In general, F will be an explicit function of t as well as of the six variables $\dot{r}$, r, G, v, H_1, θ. By making use of the canonical equations (4), we obtain

$$\frac{d}{dt}(F - T) = \frac{\partial}{\partial t}(F - T) = \frac{\partial F}{\partial t}, \quad \text{............(5)}$$

a result which is independent of the variables in terms of which F, T are expressed, provided that the equations defining any change of variables do not contain t explicitly.

If in (5) we introduce the value of T, and integrate, we obtain

$$\left(\frac{dr}{dt}\right)^2 + \frac{G^2}{r^2} = 2F - 2\int \frac{\partial F}{\partial t}\,dt. \quad \text{...............(6)}$$

The equation 1·23 (3) may be written

$$r\,\frac{d^2 r}{dt^2} - \frac{G^2}{r^2} = r\,\frac{\partial F}{\partial r}.$$

Eliminating G between this and (6) we have

$$\frac{1}{2}\,\frac{d^2}{dt^2}(r^2) = r\,\frac{\partial F}{\partial r} + 2F - 2\int \frac{\partial F}{\partial t}\,dt. \quad \text{............(7)}$$

A further useful equation may be obtained. When F is expressed as a function of r, v, Γ, θ, t, we have

$$\frac{dF}{dt} = \frac{\partial F}{\partial r}\,\dot{r} + \frac{\partial F}{\partial v}\,\dot{v} + \frac{\partial F}{\partial \Gamma}\,\dot{\Gamma} + \frac{\partial F}{\partial \theta}\,\dot{\theta} + \frac{\partial F}{\partial t}.$$

In this equation replace $\dot{v}$ by $\dot{\mathrm{v}} + \Gamma\dot{\theta}$ and substitute for $\dot{\theta}$, $\dot{\Gamma}$ their values obtained from 1·23 (5), (6). The result is

$$\frac{dF}{dt} = \frac{\partial F}{\partial r}\,\dot{r} + \frac{\partial F}{\partial v}\,\dot{\mathrm{v}} + \frac{\partial F}{\partial t}. \quad \text{..................(8)}$$

This equation when R replaces F, where $F = \mu/r + R$, is evidently still true.

1·25. *The equations of Encke and Newcomb.* Put

$$F = \frac{\mu}{r} + R, \quad \frac{d'R}{dt} = \frac{dR}{dt} - \frac{\partial R}{\partial t} = \frac{\partial R}{\partial r}\dot{r} + \frac{\partial R}{\partial v}\dot{\mathrm{v}}.$$

Equation 1·24 (7) may then be written

$$\frac{1}{2}\frac{d^2}{dt^2}(r^2) - \frac{\mu}{r} = r\frac{\partial R}{\partial r} - 2\int \frac{d'R}{dt}\,dt. \quad\text{............(1)}$$

When R is neglected, the motion becomes elliptic and r can be expressed as a periodic function of t (Chap. III). Let this value of r be denoted by r_0 and the complete value by $r_0 + \delta r$. On expanding in powers of δr and neglecting powers of δr beyond the first in the left-hand member, we obtain

$$\frac{d^2}{dt^2}(r_0\delta r) + \frac{\mu}{r_0^3}(r_0\delta r) = r\frac{\partial R}{\partial r} - 2\int \frac{d'R}{dt}\,dt. \quad\text{......(2)}$$

Elliptic values are substituted for the coordinates in the right-hand member which then becomes a function of t. The equation can then be integrated and it gives δr.

The coordinate v is obtained from

$$r^2\frac{d\mathrm{v}}{dt} = G_0 - \int \frac{\partial R}{\partial v}\,dt, \quad\text{..................(3)}$$

where G_0 is an arbitrary constant, or, neglecting terms depending on the square of the disturbing mass, from

$$\frac{d\mathrm{v}}{dt} = \frac{G_0}{r_0^2} - \frac{2G_0}{r_0^3}\delta r - \frac{1}{r_0^2}\int \frac{\partial R}{\partial v}\,dt. \quad\text{............(4)}$$

Further,

$$\frac{dv}{dt} = \frac{d\mathrm{v}}{dt} + \Gamma\frac{d\theta}{dt} = \frac{d\mathrm{v}}{dt} + \frac{\Gamma}{\frac{d\mathrm{v}}{dt}}\frac{\partial R}{\partial \Gamma} = \frac{d\mathrm{v}}{dt} + \frac{\Gamma}{\frac{dv}{dt}}\frac{\partial R}{\partial \Gamma},$$

to the same order.

If then v_0 be the value of v in elliptic motion, and $v_0 + \delta v$ its complete value, we have, to the first order of the disturbing mass,

$$\frac{d}{dt}\delta v = -\frac{2G_0}{r_0^3}\delta r - \frac{1}{r_0^2}\int \frac{\partial R}{\partial v}\,dt + \frac{\Gamma}{\dot{v}_0}\frac{\partial R}{\partial \Gamma}. \quad\text{.........(5)}$$

Since $\partial F/\partial\theta$, $\partial F/\partial\Gamma$ contain the disturbing mass as a factor, the equations 1·23 (5), (6) are immediately integrable if we put $r^2 d\mathrm{v}/dt = G_0$ in the latter.

1·26. Newcomb solves the equation for δr in the following manner. He notices that when $R=0$, the solution of the equation for r_0 contains two arbitrary constants, e the eccentricity and ϖ the longitude of perihelion (see 3·2 (7)), in addition to the arbitrary constant already present in the equation, a constant which is independent of e, ϖ. Two particular solutions of 1·25 (2) for δr with $R=0$ are obtained by varying e, ϖ. These two solutions may be written

$$r_0\delta r = r_0\frac{\partial r_0}{\partial e}, \quad r_0\delta r = r_0\frac{\partial r_0}{\partial \varpi}.$$

He then makes use of a well-known method in the solution of a linear differential equation of the second order, namely, that if $y=y_1$, $y=y_2$ are two particular solutions of the equation

$$\frac{d^2y}{dt^2} + Py = 0,$$

a particular solution of the equation

$$\frac{d^2y}{dt^2} + Py = Q$$

is
$$Cy = y_2\int Qy_1\,dt - y_1\int Qy_2\,dt,$$

where
$$y_2\frac{dy_1}{dt} - y_1\frac{dy_2}{dt} = C, \text{ a constant.}$$

In these formulae, P, Q are supposed to be known functions of t.

If X be the eccentric anomaly, we have from 3·2 (16), 3·2 (15), when X is expressed as a function of n, ϵ, e, ϖ, t,

$$\frac{\partial X}{\partial e} = -\frac{\sin X}{1-\cos X} = -\frac{a\sin X}{r}.$$

Thence, by differentiation of 3·2 (15), we obtain

$$-r_0\frac{\partial r_0}{\partial e} = a^2(\cos X - e), \quad -r_0\frac{\partial r_0}{\partial \varpi} = a^2 e\sin X.$$

Thus $\cos X - e$, $\sin X$ can be taken as the two particular solutions. They give $C=n$, the mean motion.

In the exposition of the application of these formulae to the theories of the four inner planets (*Amer. Eph. Papers*, vol. 3, pt. 5), Newcomb apparently puts $\mathrm{v}=v$, for he makes no mention of any difference between them. The difference between them, $-\int\Gamma d\theta$, which to the first order of the disturbing mass may be written $-2\sin^2\frac{1}{2}i\,\delta\theta$, is very small because the

inclinations of the orbits of these planets to the ecliptic are small. The constant and secular parts of this term are absorbed in the constants of the mean longitude, so that the only doubt which remains is whether the term gives rise to any sensible periodic terms, and if it does, whether these have been included in his final results.

1·27. *Equations of motion with the true orbital longitude as independent variable.*

These equations are deduced from those of 1·23 by making v the independent variable instead of t. The transformation is effected by the introduction of new dependent variables u, q, defined by the equations

$$u=\frac{1}{r},\quad \left(\frac{\mu}{q}\right)^{\frac{1}{2}}=G=r^2\frac{d\mathrm{v}}{dt}. \quad \text{.........(1), (2)}$$

With these definitions we have

$$\frac{dr}{dt}=\frac{dr}{d\mathrm{v}}\frac{d\mathrm{v}}{dt}=-G\frac{du}{d\mathrm{v}},$$

$$\frac{d^2r}{dt^2}=-G^2u^2\frac{d^2u}{d\mathrm{v}^2}-\frac{du}{d\mathrm{v}}\frac{dG}{dt}=-G^2u^2\left(\frac{d^2u}{d\mathrm{v}^2}+\frac{1}{G}\frac{dG}{d\mathrm{v}}\frac{du}{d\mathrm{v}}\right),$$

$$\frac{d}{dt}\left(r^2\frac{d\mathrm{v}}{dt}\right)=u^2G\frac{dG}{d\mathrm{v}},\quad \frac{d}{dt}\left(\Gamma r^2\frac{d\mathrm{v}}{dt}\right)=u^2G\frac{d}{d\mathrm{v}}(\Gamma G).$$

If then we put

$$F=\frac{\mu}{r}+\mu R=\mu(u+R), \quad \text{..................(3)}$$

so that
$$\frac{\partial F}{\partial r}=-\mu u^2-u^2\frac{\partial R}{\partial u},$$

and if we change from G to q by means of (2), the equations 1·23 (3), (4), (5), (6) are transformed to

$$\frac{d^2u}{d\mathrm{v}^2}+u-q=q\frac{\partial R}{\partial u}+\tfrac{1}{2}q\frac{dq}{d\mathrm{v}}\frac{du}{d\mathrm{v}},\quad \frac{dq}{d\mathrm{v}}=-\frac{2}{u^2}\frac{\partial R}{\partial v}, \quad \text{...(4), (5)}$$

$$\frac{dt}{d\mathrm{v}}=\frac{1}{u^2}\left(\frac{q}{\mu}\right)^{\frac{1}{2}},\quad \frac{dv}{d\mathrm{v}}=1-\Gamma\frac{d\theta}{d\mathrm{v}}, \quad \text{.........(6), (7)}$$

$$\frac{d}{d\mathrm{v}}(\Gamma q^{-\frac{1}{2}})=-\frac{q^{\frac{1}{2}}}{u^2}\frac{\partial R}{\partial \theta},\quad \frac{d\theta}{d\mathrm{v}}=\frac{q}{u^2}\frac{\partial R}{\partial \Gamma}. \quad \text{......(8), (9)}$$

It is to be noticed that the disturbing function has been denoted by μR instead of by R, so that the mass factor present in the new R is the ratio of the disturbing mass to the mass of the sun.

When we proceed by continued approximation as in Newcomb's method, the equation for u is immediately solved when that for q has been integrated, and it has the advantage of being one with constant coefficients, so that the device shown in 1·26, requiring two multiplications of series, is not needed.

When $R=0$ we have $q=\text{const.}$, and the solution of the equation for u is (3·2)

$$\frac{1}{r}=u=q\{1+e\cos(\mathrm{v}-\varpi)\},\quad 1/q=a(1-e^2).$$

Since $dq/d\mathrm{v}$ has the disturbing mass as a factor, we can change the variable u to u_1 where $u=u_1 f(q)$ without losing the easy integrability of the equation for u_1. The special cases

$$f(q)=q,\quad f(q)=q^{\frac{1}{2}}$$

have certain advantages which will be pointed out in 7·2.

It may be noticed also that when the terms in R containing the angle θ are neglected, we have $\Gamma q^{-\frac{1}{2}}=\text{const.}$, so that Γ can be completely eliminated at the outset.

The variables $1/q$, $1/u$ have the dimension of a length. If we introduce the constants n_0, a_0 such that $n_0{}^2 a_0{}^3=\mu$, and put dt_1 for $n_0 dt$, u_1 for ua_0, q_1 for qa_0, R_1 for $a_0 R$, the constants μ, a_0, n_0 will disappear from the equations and the variables are all ratios.

1·28. *Latitude equation.*

From the equations 1·23 (1), (2), namely,

$$di\sin(v-\theta)=d\theta\sin i\cos(v-\theta),\quad d\mathrm{v}=dv-(1-\cos i)\,d\theta,$$

we easily deduce

$$d\{\sin i\sin(v-\theta)\}=\sin i\cos(v-\theta)\,d\mathrm{v},$$

$$d\{\sin i\cos(v-\theta)\}=-\sin i\sin(v-\theta)\,d\mathrm{v}+\frac{\sin i\cos i}{\sin(v-\theta)}\,d\theta.$$

Whence, with the help of 1·27 (9),

$$\left(\frac{d^2}{d\mathrm{v}^2}+1\right)\sin i\sin(v-\theta)=\frac{\sin i\cos i}{\sin(v-\theta)}\frac{d\theta}{d\mathrm{v}}$$

$$=\frac{\sin i\cos i}{\sin(v-\theta)}\frac{q}{u^2}\frac{\partial R}{\partial\Gamma}\equiv Z.\quad\ldots(1)$$

It will be seen, by differentiation of 4·1 (1) with the help of the definitions of Δ, S, Γ, that $\partial R/\partial\Gamma$ contains $\sin(v-\theta)$ as a factor, so that there is no discontinuity in Z when $v-\theta$ is a multiple of π. In the section referred to, R is shown to be a function of r, r', $\cos S$ and

$$\frac{\partial R}{\partial \Gamma} = -\sin(v-\theta)\sin(v'-\theta)\frac{\partial R}{\partial \cos S}.$$

If L be the latitude of the disturbed planet above the plane of reference (1·23), equation (1) may be written

$$\left(\frac{d^2}{d\mathrm{v}^2}+1\right)\sin L = Z, \qquad \text{..................(2)}$$

so that $\sin L$ is obtained from an equation of the same type as that for u.

When Z has been expanded, we obtain v by integrating

$$\frac{dv}{d\mathrm{v}} = 1 - Z\sin i\cos i\sin(v-\theta).$$

1·29. *The Equations of motion with the disturbed Eccentric Anomaly as independent variable.*

Another variable which gives a linear form to the equation for r is X as defined by

$$r\,dX = p\,dt, \qquad p = \text{const.}$$

This variable gives

$$r\frac{dr}{dt} = p\frac{dr}{dX}, \quad \frac{d}{dt}\left(r\frac{dr}{dt}\right) = \frac{p^2}{r}\frac{d^2r}{dX^2}.$$

Equation 1·24 (7) with $F=\mu/r+R$ therefore gives

$$\frac{p^2}{r}\frac{d^2r}{dX^2} = \frac{\mu}{r} + r\frac{\partial R}{\partial r} + 2R - 2\int\frac{\partial R}{\partial t}\,dt. \qquad \text{.........(1)}$$

Define a by the equation

$$\frac{d}{dt}\left(\frac{\mu}{a}\right) = -2\frac{dR}{dt} + 2\frac{\partial R}{\partial t}\ \text{..................(2)}$$

Then equation 1·24 (6) may be written

$$\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\mathrm{v}}{dt}\right)^2 = \frac{2\mu}{r} - \frac{\mu}{a}, \qquad \text{...............(3)}$$

which shows that $2a$ is the disturbed major axis.

From (1), (2) we deduce

$$\frac{d^2r}{dX^2}+\frac{\mu}{p^2}\left(\frac{r}{a}-1\right)=\frac{r^2}{p^2}\frac{\partial R}{\partial r}. \quad\ldots\ldots\ldots\ldots\ldots(4)$$

If a_0 be an arbitrary constant, the integral of (2) may be written

$$\frac{1}{a}=\frac{1}{a_0}-2\frac{R}{\mu}+\frac{2}{\mu p}\int r\frac{\partial R}{\partial t}dX, \quad\ldots\ldots\ldots\ldots\ldots(5)$$

and, with the aid of this equation, (4) becomes

$$\frac{d^2r}{dX^2}+\frac{\mu}{p^2a_0}(r-a_0)=\frac{r^2}{p^2}\frac{\partial R}{\partial r}+\frac{\mu r}{p^2}\left(\frac{1}{a}-\frac{1}{a_0}\right). \quad\ldots\ldots(6)$$

The transformation of the remaining equations to the variable X as independent variable is effected immediately.

If n_0, p be defined by the equations $\mu p^2=a_0$, $\mu=n_0^2a_0^3$, the definition of X gives $r\,dX=a_0n_0\,dt$. A reference to Chap. III shows that in undisturbed motion, X is the eccentric anomaly.

If the equation (6) be solved by the method outlined in 1·26, it will be seen that the solution is closely analogous to that of Newcomb, when we change the variable from t to X under the integral sign. It has, however, the advantage of being exact instead of approximate and is thus adaptable to the calculation of the higher approximations.

These equations, which appear to be new, will not be developed further in this volume. The general method of treatment would follow lines similar to those adopted when the true longitude is taken as the independent variable (Chap. VII). It may, however, be noticed that, since

$$r=a(1-e\cos X)$$

in elliptic motion, the equations are integrated without multiplications of series when R, $r\partial R/\partial r$ have been expressed in terms of X. The only exception is the equation for v—an exception common to all methods.

1·30. *Equations of motion referred to the coordinates of the disturbing planet.*

Put $$r=r'\rho, \quad r'^2\frac{dv'}{dt}=h', \quad \rho^2\frac{d\text{v}}{dv'}=h_\rho, \quad\ldots\ldots\ldots\ldots(1)$$

so that $$r^2\frac{d\text{v}}{dt}=h'\rho^2\frac{d\text{v}}{dv'}=h'h_\rho. \quad\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

A direct transformation from the variables r, t to ρ, v' gives

$$\frac{d^2r}{dt^2}-r\left(\frac{d\text{v}}{dt}\right)^2=\frac{h'^2}{r'^3}\left\{\frac{d^2\rho}{dv'^2}-\rho\left(\frac{d\text{v}}{dv'}\right)^2\right\}+\frac{1}{r'}\frac{dh'}{dt}\frac{d\rho}{dv'}+\rho\frac{d^2r'}{dt^2}.$$

This equation, with the aid of the definitions

$$F=\frac{F_1}{r'},\quad \frac{\partial F}{\partial r}=\frac{1}{r'^2}\frac{\partial F_1}{\partial \rho}, \quad\text{.................(3)}$$

enables us to transform 1·23 (3) to

$$\frac{d^2\rho}{dv'^2}-\rho\left(\frac{d\mathrm{v}}{dv'}\right)^2=\frac{r'}{h'^2}\frac{\partial F_1}{\partial \rho}+\rho\frac{r'^3}{h'^2}\frac{d^2r'}{dt^2}-\frac{r'^2}{h'}\frac{dh'}{dt}\frac{d\rho}{dv'}\ldots\text{(4)}$$

With the definitions (1) of h_ρ, h', equation 1·23 (4) gives

$$h'\frac{dh_\rho}{dt}+h_\rho\frac{dh'}{dt}=\frac{\partial F}{\partial v}.$$

Replacing dt by r'^2dv'/h' and F by F_1/r', we obtain

$$\frac{dh_\rho}{dv'}=\frac{r'}{h'^2}\frac{\partial F_1}{\partial v}-\frac{r'^2}{h'}\frac{dh'}{dt}h_\rho.\quad\text{.................(5)}$$

Now let r', v' be the polar coordinates of a disturbing planet moving in the plane of reference and satisfying the equations

$$\frac{d^2r'}{dt^2}-r'\left(\frac{dv'}{dt}\right)^2=\frac{\partial F'}{\partial r'},\quad \frac{dh'}{dt}=\frac{\partial F'}{\partial v'}.\quad\text{......(6), (7)}$$

The elimination of d^2r'/dt^2, dh'/dt from (4), (5) by means of (6), (7) gives

$$\begin{aligned}\frac{d^2\rho}{dv'^2}-\rho\left(\frac{dv}{dv'}\right)^2&=\frac{r'}{h'^2}\frac{\partial F_1}{\partial \rho}+\rho\left(1+\frac{r'^3}{h'^2}\frac{\partial F'}{\partial r'}\right)-\frac{r'^2}{h'}\frac{dh'}{dt}\frac{d\rho}{dv'}\\&=\frac{\partial}{\partial\rho}\left\{\frac{r'}{h'^2}F_1+\frac{\rho^2}{2}\left(1+\frac{r'^3}{h'^2}\frac{\partial F'}{\partial r'}\right)\right\}-\frac{r'^2}{h'}\frac{dh'}{dt}\frac{d\rho}{dv'},\end{aligned}\quad\text{......(8)}$$

$$\frac{dh_\rho}{dv'}=\frac{r'}{h'^2}\frac{\partial F_1}{\partial v}-\frac{r'^2}{h'}\frac{\partial F'}{\partial v'}.\quad\text{.........................(9)}$$

The transformation of the equations for Γ, θ is easily made.

When the disturbing planet moves in an ellipse, we have (Chap. III)

$$\frac{r'}{h'^2}=\frac{1}{\mu'\{1+e'\cos(v'-\varpi')\}},\quad \frac{\partial F'}{\partial r'}=-\frac{\mu'}{r'^2},\quad \frac{\partial F'}{\partial v'}=0,$$

so that the equations become

$$\frac{d^2\rho}{dv'^2}-\rho\left(\frac{d\mathrm{v}}{dv'}\right)^2=\frac{\partial F_2}{\partial\rho},\quad \frac{d}{dv'}\left(\rho^2\frac{d\mathrm{v}}{dv'}\right)=\frac{\partial F_2}{\partial v},\quad\text{.....(10)}$$

where $$F_2 = \frac{F_1 + \frac{1}{2}\rho^2 e' \cos(v' - \varpi')}{1 + e' \cos(v' - \varpi')}, \quad \text{...........(11)}$$

equations which have a form similar to that which obtains when t is the independent variable.

When the motion of the disturbing planet is no longer elliptic, we put

$$F' = \frac{\mu'}{r'} + R'.$$

The additional terms in (10) are easily written down: they contain the derivatives $\partial R'/\partial r'$, $\partial R'/\partial v'$.

The chief point of interest in this form of the equations of motion arises from the fact that F_1 is independent of r', t explicitly. That this is the case is seen from 1·10 when we put

$$v_k = v, \quad v_j = v', \quad \theta_j = \theta, \quad \Gamma_j = \Gamma, \quad m_0 + m_j = \mu, \quad m_k = m'.$$

For then $$F_1 = Fr' = \frac{\mu}{\rho} + m' \left\{ \frac{1}{(1 + \rho^2 - 2\rho \cos S)^{\frac{1}{2}}} - \rho \cos S \right\},$$

where $$\cos S = (1 - \tfrac{1}{2}\Gamma) \cos (v - v') + \tfrac{1}{2}\Gamma \cos (v + v' - 2\theta).$$

Thus the only way in which the eccentricity of the disturbing planet appears in the equations of motion is in the explicit form shown in F_2. It should be pointed out, however, that if we put $m' = 0$ so that F_1 is reduced to the term m/ρ, the solution of the equations for ρ, v will contain e'; hence, in the second approximation, when we substitute these values in the coefficient of m', the development of F_1 will contain e'. Nevertheless the equations appear to have possibilities for usefulness in the discussion of certain problems in which it is necessary to take into account the perturbations of the disturbing planet by another planet—'the indirect effect' of the latter, that is, the effect transmitted through the disturbing planet. A case of this kind is the effect of the action of Saturn on Jupiter where the latter is disturbing the motion of an asteroid of the Trojan group. Another case is the indirect effect of a planet on the motion of the moon.

1·31. The most important of the various forms of the equations of motion, namely, that referred to the 'elliptic frame,' will be developed in Chap. v. In this form the two variables r, v are replaced by four new variables which are so chosen that their *first* derivatives only appear in the equations of motion. The elliptic frame in various forms has been extensively used for the calculation of perturbations. In its direct form it was used by Leverrier for obtaining the orbits of the major planets and in the 'canonical' form by Delaunay for the motion of the moon. In a different form it was used by Hansen for both the planets and the moon.

Another group of methods depends on the use of a uniformly rotating frame of reference. This, like most of the methods which have been used for actual calculation, was initiated by Euler and adopted by G. W. Hill and E. W. Brown for the development of the motion of the moon as disturbed by the sun.

For these and other methods not treated here, the reader is referred to standard treatises like those of Tisserand and Brown. References up to the date of their publication will be found in the articles dealing with celestial mechanics in the *Ency. Math. Wiss.*

1·32. *Motion referred to an arbitrary plane of reference.*

In the preceding developments the plane of reference has been that of the motion of the disturbing planet.

Now let the symbols v, θ, i refer to any fixed arbitrary reference plane and let v', θ', i' have the corresponding significations for the disturbing planet; the rectangular coordinates of the disturbed planet are then given by 1·23 (7) and similar formulae will hold for the disturbing planet.

We have $rr' \cos S = xx' + yy' + zz'$.

Put $\iota = \sqrt{-1}$ so that

$$xx' + yy' \text{ is the real part of } (x + \iota y)(x' - \iota y').$$

Define Γ, k, Γ', k' by

$$\Gamma = 1 - \cos i, \quad 2k = 1 + \cos i, \quad \Gamma' = 1 - \cos i', \quad 2k' = 1 + \cos i'.$$

We can then obtain from 1·23 (7),

$$\frac{x + \iota y}{r} = ke^{v\iota} + \tfrac{1}{2}\Gamma e^{(-v+2\theta)\iota},$$

and similarly

$$\frac{x' - \iota y'}{r'} = k'e^{-v'\iota} + \tfrac{1}{2}\Gamma' e^{(v'-2\theta')\iota}.$$

The product contains v, v' only in the combinations $v \mp v'$. If we form it and, after taking the real part, separate the coefficients of the sines and cosines of these two angles we obtain

$$\begin{aligned} &\cos(v - v')\{kk' + \tfrac{1}{4}\Gamma\Gamma' \cos(2\theta - 2\theta')\} \\ &+ \sin(v - v')\{\tfrac{1}{4}\Gamma\Gamma' \sin(2\theta - 2\theta')\} \\ &+ \cos(v + v')\{\tfrac{1}{2}k\Gamma' \cos 2\theta' + \tfrac{1}{2}k'\Gamma \cos 2\theta\} \\ &+ \sin(v + v')\{\tfrac{1}{2}k\Gamma' \sin 2\theta' + \tfrac{1}{2}k'\Gamma \sin 2\theta\}. \end{aligned}$$

To find $\cos S$ we must add zz'/rr' to this. The latter can be written

$$\tfrac{1}{2}\sin i \sin i' \{\cos(v-v')\cos(\theta-\theta') + \sin(v-v')\sin(\theta-\theta') \\ - \cos(v+v')\cos(\theta+\theta') - \sin(v+v')\sin(\theta+\theta')\}.$$

We can therefore express $\cos S$ in the form

$$\cos S = K_0 \cos(v - v' - K_1) + K_2 \cos(v + v' - K_3),$$

where $K_0 = 1$, $K_1 = K_2 = K_3 = 0$ when $i = i' = 0$. Since the K's are functions of θ, θ', i, i', only, $\cos S$ has the same form as in the simpler case. Here

$$\begin{array}{lllll} & K_0, & \varpi - \varpi' - K_1, & K_2, & K_3 \\ \text{replace} & 1 - \tfrac{1}{2}\Gamma, & \varpi - \varpi', & \tfrac{1}{2}\Gamma, & 2\theta, \end{array}$$

used in the developments of the later chapters.

In these and many similar cases we replace

$$A\cos\alpha + B\sin\alpha \quad \text{by} \quad C\cos(\alpha - \alpha_0),$$

where C, α_0 are determined from

$$C\cos\alpha_0 = A, \quad C\sin\alpha_0 = B,$$

C being in general so taken as to have a positive sign.

CHAPTER II

METHODS FOR THE EXPANSION OF A FUNCTION

2·1. The greater part of the work of solving any problem in celestial mechanics consists of the expansions of various functions into sums of periodic terms, mainly because the integrals of these functions cannot be obtained conveniently in any other way. The majority of these methods, which depend chiefly on Taylor's expansion and Fourier's theorem, are well known, but there are certain expansions, continually recurring, which require much labour. It is the purpose of this chapter to ease the work, partly by giving formulae which are ready for immediate application, and partly by so arranging them that the calculations may be carried out with the least chance of error. Certain of the formulae are intended to be used only when literal expansions are required: when the coefficients are numerical the methods of harmonic analysis usually give higher accuracy and are less laborious.

The coefficient of a periodic term in the expansions of most of the functions considered here takes the form

$$\alpha^i (a_0 + a_1\alpha^2 + a_2\alpha^4 + \dots), \quad \text{..................(1)}$$

where α is a parameter and $a_0, a_1, a_2, \dots$ are integers or fractions. It is frequently required to calculate the function as far as some definite power of α, and to carry one or two coefficients considerably further. It is this latter need which causes difficulty because there is much wasted labour if the whole series be carried to this higher power. This fact has to be remembered when a choice of any method of expansion is made.

Expansions in power series are so much easier to perform and are so much less subject to error than operations with series of periodic functions, that the latter are usually reduced to the former by the substitutions

$$x = \exp. \theta \sqrt{-1}, \quad 2 \cos i\theta = x^i + x^{-i}, \quad 2\sqrt{-1} \sin i\theta = x^i - x^{-i}. \quad \text{......(2)}$$

When the coefficient of $\cos i\theta$ or $\sin i\theta$ has the form (1) the work is made easier by the substitutions

$$\left.\begin{array}{ll} z = a \exp.\theta\sqrt{-1}, & p = a^2, \\ 2a^i \cos i\theta = z^i + p^i z^{-i}, & 2a^i\sqrt{-1}\sin i\theta = z^i - p^i z^{-i}. \end{array}\right\} \quad \ldots(3)$$

We then expand in positive powers of p and in positive and negative powers of z. This simple change from the substitution (2) not only gives greater freedom in the choice of methods of expansion, but aids materially in solving the problem referred to on the preceding page.

Extensive use is made of another device, namely, that of expansions of functions of an operator. These usually take the form $\phi(D).f(x)$ where $D = d/dx$. It is then always supposed that $\phi(D)$ is developable in the form

$$\phi(D) = a_0 + a_1 D + a_2 D^2 + \ldots,$$

where $a_0, a_1, a_2, \ldots$ are independent of x, so that

$$\phi(D).f(x) = a_0 f + a_1 \frac{df}{dx} + a_2 \frac{d^2 f}{dx^2} + \ldots.$$

Operations with functions of D are performed in accordance with the rules of ordinary algebra except that functions of D and those of x do not follow the commutative law of multiplication. The gain is partly in brevity of expression and partly in the methods of expansion which are suggested by well-known expansions. Thus we can use such forms as $\exp. D$, $\log(1 + D)$, $(1 + D)^n$, $(1 + a)^D$, $\cos D$, etc., each acting on $f(x)$.

2·2. *Lagrange's theorem for the expansion of a function defined by an implicit equation.* Let the equation be

$$y = x + \alpha\phi(y) = x + \alpha\phi, \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

where α is a parameter and ϕ and its derivatives are continuous functions of y. The problem is the expansion of $F(y) = F$ in powers of α with coefficients which are functions of x. The theorem gives

$$F(y) = F_x + \alpha\left(\phi_x \frac{dF_x}{dx}\right) + \frac{\alpha^2}{2!}\frac{d}{dx}\left(\phi_x{}^2 \frac{dF_x}{dx}\right) + \frac{\alpha^3}{3!}\frac{d^2}{dx^2}\left(\phi_x{}^3 \frac{dF_x}{dx}\right) + \ldots, \quad \ldots\ldots(2)$$

where $$F_x = F(x), \quad \phi_x = \phi(x).$$

The proof which follows indicates an extension of the theorem to several variables.

We may regard F as a function of α, x. Regarded as a function of α, it may be expanded in powers of α by Maclaurin's theorem in the form

$$F = F_0 + \alpha (AF)_0 + \frac{\alpha^2}{2!}(A^2F)_0 + \ldots, \quad A = \frac{\partial}{\partial \alpha}, \quad \ldots\ldots(3)$$

where the zero suffix denotes that α is put equal to zero after the derivatives have been formed. Evidently $F_0 = F_x$. Put $D = \partial/\partial x$. The use of the operators A, D implies that the functions on which they operate are expressed as functions of x, α.

Operating on (1) with A, D, successively, we have

$$Ay = \phi + \alpha \frac{d\phi}{dy} Ay, \quad Dy = 1 + \alpha \frac{d\phi}{dy} Dy,$$

so that $$Ay = \phi\, Dy,$$

and therefore, for any function g of y,

$$Ag = \frac{dg}{dy} Ay = \frac{dg}{dy} \phi\, Dy = \phi\, Dg. \quad \ldots\ldots\ldots\ldots(4)$$

From this result we can show, by induction, that

$$A^nF = D^{n-1}(\phi^n DF). \quad \ldots\ldots\ldots\ldots\ldots\ldots(5)$$

Assume that (5) is true and operate on it with A. Then, since x, α are independent so that D, A are commutative,

$$\begin{aligned} A^{n+1}F &= D^{n-1}\{DF \,.\, A\phi^n + \phi^n A(DF)\} \\ &= D^{n-1}\{DF \,.\, \phi D\phi^n + \phi^n D(AF)\}, \end{aligned}$$

the change in the first term being made by the use of (4) and in the second by the commutation of A, D. But, by putting $g = F$ in (4) and operating with D we have $D(AF) = D(\phi DF)$, so that the portion under the operator D^{n-1} is $D(\phi DF \,.\, \phi^n)$. Hence

$$A^{n+1}F = D^n(\phi^{n+1} DF),$$

and since the theorem is true for $n = 1$, it holds universally.

Finally, since ϕ, F become ϕ_x, F_x when $\alpha=0$, the coefficient of α^n in (3) becomes the same as that in (2), and the theorem is proved.

Particular case. When $F(y)=y$, we have

$$y = x + \alpha\phi_x + \frac{\alpha^2}{2!}\frac{d}{dx}(\phi_x^2) + \frac{\alpha^3}{3!}\frac{d^2}{dx^2}(\phi_x^3) + \dots \quad \dots\dots(6)$$

2·3. *Extension of Lagrange's theorem.*

If $y_i = x_i + \alpha a_i \phi(y_1, y_2, y_3) = x_i + \alpha a_i \phi, \quad i = 1, 2, 3, \quad \dots(1)$

where the a_i are constants and α is a parameter, and if

$$F = F(y_1, y_2, y_3),$$

where F and ϕ are continuous functions with continuous derivatives of y_1, y_2, y_3, and further, if

$$F_x = F(x_1, x_2, x_3), \quad \phi_x = \phi(x_1, x_2, x_3),$$

$$D^n = a_1\left(\frac{\partial}{\partial x_1}\right)^n + a_2\left(\frac{\partial}{\partial x_2}\right)^n + a_3\left(\frac{\partial}{\partial x_3}\right)^n,$$

then

$$F(y_1, y_2, y_3) = F_x + \alpha\phi_x DF_x + \frac{\alpha^2}{2!}D(\phi_x^2 DF_x) + \frac{\alpha^3}{3!}D^2(\phi_x^3 DF_x). \quad \dots\dots(2)$$

The proof follows the same general lines as before. We first regard F as a function of α and expand in the form 2·2 (3). Next, by differentiation of (1),

$$\frac{\partial y_i}{\partial x_j} = \frac{1}{0} + \alpha a_i \Sigma_k \frac{\partial\phi}{\partial y_k}\frac{\partial y_k}{\partial x_j}, \qquad k = 1, 2, 3, \quad \dots\dots(3)$$

where the first term of the right-hand member is 1 or 0 according as $i=j$ or $i \neq j$. This equation is multiplied by a_j and summed for $j=1, 2, 3$. The result is

$$Dy_i = a_i + \alpha a_i \Sigma \frac{\partial\phi}{\partial y_k} Dy_k. \quad \dots\dots\dots\dots\dots(4)$$

But from (1) we have

$$\frac{\partial y_i}{\partial\alpha} = a_i\phi + \alpha a_i \Sigma \frac{\partial\phi}{\partial y_k}\frac{\partial y_k}{\partial\alpha}. \quad \dots\dots\dots\dots\dots(5)$$

These two sets of equations may be regarded as linear, the first set for the determination of Dy_i, and the second for that of $\partial y_i/\partial\alpha$. They are the same except that the absolute terms in the latter are ϕ times those in the former. Hence

$$\frac{\partial y_i}{\partial\alpha}=\phi D y_i,$$

and therefore
$$\frac{\partial F}{\partial\alpha}=\phi DF.$$

The remainder of the proof is the same as that in 2·2.

2·4. The most general case in which

$$y_i=x_i+\alpha\phi_i(y_1, y_2, y_3), \qquad i=1, 2, 3,$$

does not seem to be soluble by a simple general formula. It is not difficult, however, to obtain the solution as far as α^2. If we put

$$\phi_{ix}=\phi_i(x_1, x_2, x_3), \quad D=\Sigma_i\phi_{ix}\frac{\partial}{\partial x_i}, \quad F_x=F(x_1, x_2, x_3),$$

we obtain

$$F(y_1, y_2, y_3)=F_x+\alpha DF_x+\frac{\alpha^2}{2}\Sigma_j\left(\frac{\partial F_x}{\partial x_j}D\phi_{jx}+D\,.\,DF_x\right).$$

Lagrange's theorem can be used to find X or any function of X in terms of g from Kepler's equation $X=g-e\sin X$ (3·2 (16)), and was probably suggested by this problem. The extension may be applied to the Jacobian solution of the canonical equations (Chap. v) to find the new variables in terms of the old or vice versa, when the disturbing function is confined to a single periodic term or to a Fourier set of terms. The more general case mentioned above is that of the Jacobian solution where the disturbing function contains any periodic terms.

2·5. *Transformation of a Fourier expansion with argument y into one with argument x, where y is defined in terms of x by means of an implicit equation*.*

Let $F(y)$ be expanded in the form

$$F(y)=\Sigma(c_i\cos iy+d_i\sin iy), \qquad i=0, 1, 2, \ldots \quad \ldots(1)$$

and let
$$y=x+\alpha\phi(y), \quad \ldots\ldots\ldots\ldots(2)$$

where α is a parameter and $\phi(y)$ is expressed in the same form

* E. W. Brown, *Proc. Nat. Acad. Sc. Wash.* vol. 16 (1930), p. 150.

as $F(y)$. It is required that we obtain the coefficients a_i, b_i, when $F(y)$ is expressed in the form

$$F = \Sigma\,(a_i \cos ix + b_i \sin ix). \qquad\text{.............. (3)}$$

With the help of the notation,

$$\phi = \phi\,(x), \quad F = F(x), \quad D = d/dx,$$

and with the use of Lagrange's theorem, (2) gives

$$F(y) = F + \Sigma \frac{\alpha^n}{n!} D^{n-1}(\phi^n DF), \qquad n = 1, 2, \ldots\ldots \qquad\text{...(4)}$$

Let ψ be another function of x expressed as a Fourier series. On multiplying both members of (4) by $D\psi$, we obtain

$$F(y)\,.\,D\psi = FD\psi + \Sigma \frac{\alpha^n}{n!} D\psi\,.\,D^{n-1}(\phi^n DF). \qquad\text{...(5)}$$

The identity,

$$D\psi\,.\,D^{n-1}(\phi^n DF) = D\,\{D\psi\,.\,D^{n-2}(\phi^n DF)\} - D^2\psi\,.\,D^{n-2}(\phi^n DF),$$

shows that, when all three terms are expressed as Fourier series, the constant term of the left-hand member is the same as that of the last term, the remaining term being the derivative of a Fourier series. By repeating this process $n-2$ times, we deduce the fact that the constant terms in the Fourier expansions of

$$D\psi\,.\,D^{n-1}(\phi^n DF), \quad (-1)^{n-1} D^n\psi\,.\,\phi^n DF$$

are the same. On applying this result to each term of the right-hand member of (5) we obtain a series which is the expansion, by Taylor's theorem, of

$$-DF\,.\,\psi(x - \alpha\phi) \qquad\text{.................... (6)}$$

in powers of $\alpha\phi$.

Hence, *the constant term in the Fourier expansion of*

$$F(y)\frac{d}{dx}\psi(x), \quad y = x + \alpha\phi(y), \qquad\text{........... (7)}$$

when $F(y)$ is expressed in terms of x, is the same as the constant term in the Fourier expansion of

$$-\frac{d}{dx}F(x)\,.\,\psi\,\{x - \alpha\phi(x)\}, \qquad\text{.............. (8)}$$

or in that of

$$F(x)\,.\,\frac{d}{dx}\psi\,\{x - \alpha\phi(x)\}. \qquad\text{.............. (9)}$$

It is to be remembered that $\psi\{x-\alpha\phi(x)\}$ means the result obtained by replacing x by $x-\alpha\phi(x)$ in $\psi(x)$. We obtain (9) from (8) by noticing that their difference is the derivative of the product of the two functions $F(x)$, $\psi\{x-\alpha\phi(x)\}$.

The use of (9) enables us to state the theorem in a slightly different form. If we put $\chi(x)$ for $d\psi(x)/dx$, we have the theorem:

The constant term in the Fourier expansion of

$$F(y)\chi(x), \quad y=x+\alpha\phi(y), \qquad \text{...........(10)}$$

where $F(y)$ is expressed in terms of x, is the same as the constant term in the Fourier expansion of

$$F(x)\chi(z)\frac{dz}{dx}, \quad z=x-\alpha\phi(x). \qquad \text{.........(11)}$$

According to this definition, χ contains no constant term. But if χ is a constant, (4) shows that the constant term in $F(y)$ is the same as that in $F+\alpha\phi\,.\,DF$, which is the same as that in $F(1-\alpha D\phi)$, so that the theorem still holds when $\chi(x)$ contains a constant term.

Since we are concerned only with the constant terms in (8), (9) or (11), the theorem evidently holds if we replace the letter x by the letter y in these three formulae.

The chief value of this theorem lies in the fact that it removes the necessity for solving the implicit equation $y=x+\alpha\phi(y)$ in order to get y in terms of x.

The application to the coefficients in (3) is immediate. If we put $i\psi=\sin ix$, so that $D\psi=\cos ix$, and note that the constant term in the product of $D\psi$ by the right-hand member of (3) is $\frac{1}{2}a_i$, we find from (8) the result,

$$a_i=\text{constant term in } -\frac{2}{i}\sin\{ix-i\alpha\phi(x)\}\frac{d}{dx}F(x). \qquad \text{......(12)}$$

Similarly, by taking $D\psi=\sin ix$, we obtain

$$b_i=\text{constant term in } \frac{2}{i}\cos\{ix-i\alpha\phi(x)\}\frac{d}{dx}F(x), \qquad \text{...(13)}$$

and from (9)

$$a_0=\text{constant term in } \left\{1-\alpha\frac{d}{dx}\phi(x)\right\}F(x). \qquad \text{...(14)}$$

When y, x take the values 0, π together, these last results may be deduced from a change of variable in the Fourier integral.

2·6. *Extension to two variables.* Suppose that we have a second pair of variables x', y', independent of x, y, connected by the implicit relation $y' = x' + \alpha' \phi'(y')$ and that we desire to obtain the expansion of $F(y, y')$ in the form

$$\Sigma_{i,i'} \{a_{ii'} \cos(ix + i'x') + b_{ii'} \sin(ix + i'x')\}, \quad \ldots\ldots(1)$$

where i, i' are positive and negative integers. An expansion in this form will be called a double Fourier series.

For the development we adopt a notation similar to that used before, namely,

$$F = F(x, x'), \quad D = \partial/\partial x, \quad D' = \partial/\partial x', \text{ etc.}$$

A double application of Lagrange's theorem gives

$$F(y, y') = \Sigma_{n,m} \frac{\alpha^n \alpha'^m}{n!m!} D^{n-1} D'^{m-1} \{\phi^n \phi'^m DD'F\},$$

where the signification when $n = 0$ or $m = 0$ is the same as that shown by 2·5 (4). This is multiplied by $DD'\psi(x, x')$, where ψ is a double Fourier series and the process adopted in 2·5 is then followed for each of the variables x, x'. It evidently leads to similar results which can be stated in the following theorems.

The constant term in the double Fourier expansion of

$$F(y, y') \frac{\partial^2}{\partial x \partial x'} \psi(x, x'), \quad y = x + \alpha\phi(y), \quad y' = x' + \alpha'\phi'(y'),$$

when $F(y, y')$ is expressed in terms of x, x', is the same as the constant term in the double Fourier expansion of

$$\frac{\partial^2}{\partial x \partial x'} F(x, x') \psi\{x - \alpha\phi(x),\ x' - \alpha'\phi'(x')\}, \quad \ldots\ldots(2)$$

or in that of

$$F(x, x') \frac{\partial^2}{\partial x \partial x'} \psi\{x - \alpha\phi(x),\ x' - \alpha'\phi'(x')\}. \quad \ldots\ldots(3)$$

And the constant term in the double Fourier expansion of

$$F(y, y')\,\chi(x, x') \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

is the same as that in

$$F(x, x')\,\chi(z, z')\,\frac{dz}{dx}\cdot\frac{dz'}{dx'},\quad z = x - \alpha\phi(x),\quad z' = x' - \alpha'\phi'(x'). \qquad \text{......(5)}$$

It will be noticed that the double operation removes the negative sign present in 2·5 (8). It reappears, however, if we replace (5) by

$$-\frac{\partial}{\partial x}F(x, x')\,\frac{\partial}{\partial x'}\,\psi\,\{x - \alpha\phi(x),\ x' - \alpha'\phi'(x')\}, \qquad \text{...(6)}$$

or by the similar formula in which the derivatives $\partial/\partial x$, $\partial/\partial x'$ are interchanged.

In order to apply the theorem to the coefficients in (1), we put $DD'\psi$ equal to $\cos(ix + i'x')$, $\sin(ix + i'x')$, successively in (2). We obtain

$$\begin{matrix} a_{ii'} \\ b_{ii'} \end{matrix} = \text{const. term in} \begin{matrix} -\cos \\ -\sin \end{matrix}\{ix + i'x' - i\alpha\phi(x) - i'\alpha'\phi'(x')\}$$

$$\times\,\frac{2}{ii'}\,\frac{\partial^2}{\partial x\,\partial x'}F(x, x'), \qquad \text{......(7)}$$

which hold only when i, i' are both different from zero.

When $i = 0$, we use (5) with $\chi(x, x')$ equal to $\cos i'x'$, $\sin i'x'$, successively, and obtain

$$\begin{matrix} a_{0i'} \\ b_{0i'} \end{matrix} = \text{const. term in} \begin{matrix} +\cos \\ +\sin \end{matrix}\{i'x' - i'\alpha'\phi'(x')\}$$

$$\times\left\{1 - \alpha\frac{\partial}{\partial x}\phi(x)\right\}\left\{1 - \alpha'\frac{\partial}{\partial x'}\phi'(x')\right\}F(x, x'), \qquad \text{......(8)}$$

$$\text{or in}\quad \begin{matrix} -\sin \\ +\cos \end{matrix}\{i'x' - i'\alpha'\phi'(x')\}\left\{1 - \alpha\frac{\partial}{\partial x}\phi(x)\right\}\frac{2}{i'}\frac{\partial}{\partial x'}F(x, x'). \qquad \text{......(9)}$$

The formulae for a_{i0}, b_{i0} are similar.

When $i = i' = 0$, we use (5) with $\chi(x, x') = 1$, and obtain

$$a_{00} = \text{const. term in}\left\{1 - \alpha\frac{\partial}{\partial x}\phi(x)\right\}\left\{1 - \alpha'\frac{\partial}{\partial x'}\phi'(x')\right\}F(x, x'). \qquad \text{......(10)}$$

2·7. *Expansion by symbolic operators.*

If p^D is expansible in positive integral powers of D and if $f(x)$ is expansible in integral powers of x, then

$$f(px) = p^D f(x), \text{ where } D = x\frac{d}{dx} = \frac{d}{d\log x}.$$

If m be a positive integer, we have

$$D^m(x^n) = n^m x^n.$$

Hence, if
$$p^D = a_0 + a_1 D + a_2 D^2 + \ldots,$$
we have
$$p^D x^n = (px)^n.$$

If then
$$f(x) = \Sigma_n(b_n x^n), \qquad n = 0, \pm 1, \ldots,$$
we have
$$f(px) = \Sigma_n b_n (px)^n = p^D \Sigma_n b_n x^n = p^D f(x).$$

This theorem can evidently be extended to any number of variables. Thus

$$f(p_1 x_1, p_2 x_2, \ldots) = p_1^{D_1} . p_2^{D_2} \ldots f(x_1, x_2, \ldots), \text{ where } D_i = x_i \partial/\partial x_i.$$

The application of the theorem depends on the possibility of expanding p^D in powers of D in such a manner that we can, by stopping at some definite power, secure a given degree of accuracy. This happens when p has the form $(1+\epsilon y)^k$, where ϵ is a small parameter such that ϵy is less than unity. We can then use the binomial theorem and obtain

$$p^D = (1+\epsilon y)^{kD} = 1 + \epsilon y . kD + \frac{\epsilon^2 y^2}{1.2} kD(kD-1) + \ldots.$$

We can also make use of the expansion of e^z in powers of z if we put $z = D\log p$, for then

$$p^D = 1 + \log p . D + \frac{(\log p)^2}{1.2} . D^2 + \ldots,$$

and $\log p$ has the factor ϵ.

2·8. *Product of two Fourier series.*

(i) Let the series be

$$A = a_0 + 2\Sigma a_i \alpha^i \cos i\theta, \quad B = b_0 + 2\Sigma b_i \alpha^i \cos i\theta, \qquad i = 1, 2, \ldots.$$

Put

$$z = \alpha \exp . \theta\sqrt{-1}, \quad p = \alpha^2, \quad 2\alpha^i \cos i\theta = z^i + p^i z^{-i},$$

$$A_0 = a_0 + \Sigma a_i z^i, \quad B_0 = b_0 + \Sigma b_i z^i. \ \ldots\ldots\ldots\ldots(1)$$

Then $$AB = (A_0 + \Sigma a_i p^i z^{-i})(B_0 + \Sigma b_i p^i z^{-i}). \quad\text{.........(2)}$$

Since A, B are even functions of θ, their product will have the same property and the coefficient of z^i will be the same as that of $p^i z^{-i}$. It is therefore sufficient to find the coefficient of z^i, $i > 0$, and then to replace z^i by $2\alpha^i \cos i\theta$.

We thus reject all negative powers of z in (2) and it is therefore necessary only to find the coefficient of z^i in

$$A_0 B_0 + A_0 \Sigma b_i p^i z^{-i} + B_0 \Sigma a_i p^i z^{-i}.$$

The first term is the product of the power series (1) and from this product we select the coefficient of z^i. In the second term we select the coefficient of $p^j z^i$, namely, $a_{i+j} b_j$, and sum for $j = 1, 2, \ldots$; the third term is treated in a similar manner.

On performing these operations and replacing p by α^2, we find for the coefficient of z^i, that is, of $2\alpha^i \cos i\theta$

$$(a_0 b_i + a_1 b_{i-1} + \ldots + a_i b_0) + \alpha^2 (a_1 b_{i+1} + a_{i+1} b_1)$$
$$+ \alpha^4 (a_2 b_{i+2} + a_{i+2} b_2) + \ldots, \quad\text{......(3)}$$

and for the constant term,

$$a_0 b_0 + 2\alpha^2 a_1 b_1 + 2\alpha^4 a_2 b_2 + \ldots. \quad\text{...............(4)}$$

The parameter α may not be present and we then put $\alpha = 1$. In the series with which we have to deal it is usually present implicitly, if not explicitly, so that the order of magnitude of any coefficient is denoted by its suffix, and in the product by the sum of the suffixes of a, b. Thus the arrangement in (3) is made as needed, namely, with respect to the orders of the terms, the brackets giving successively the terms of orders $i, i+2, i+4, \ldots$.

(ii) If
$$A' = 2\Sigma a_i \alpha^i \sin i\theta, \quad B' = 2\Sigma b_i \alpha^i \sin i\theta,$$
we adopt the same substitution for θ, α and we have

$$2\sqrt{-1}\,\alpha^i \sin i\theta = z^i - p^i z^{-i}.$$

The product is an even function of θ and is therefore expressed in terms of cosines of $i\theta$. We have only to find the coefficient of z^i in

$$-A_0 B_0 + A_0 \Sigma b_i p^i z^{-i} + B_0 \Sigma a_i p^i z^{-i},$$

in which $a_0 = b_0 = 0$. Hence we obtain, for the coefficient of $2\alpha^i \cos i\theta$,

$$-(a_1 b_{i-1} + a_2 b_{i-2} + \ldots + a_{i-1} b_1) + \alpha^2 (a_1 b_{i+1} + a_{i+1} b_1) + \alpha^4 (a_2 b_{i+2} + a_{i+2} b_2) + \ldots \quad \ldots\ldots\ldots(5)$$

and, for the constant term,

$$2\alpha^2 a_1 b_1 + 2\alpha^4 a_2 b_2 + \ldots \quad \ldots\ldots\ldots\ldots\ldots\ldots(6)$$

(iii) If

$$A' = 2\Sigma a_i \alpha^i \sin i\theta, \quad B = b_0 + 2\Sigma b_i \alpha^i \cos i\theta,$$

the product is an odd function of θ. Here we need only the coefficient of $z^i \div \sqrt{-1}$ in

$$A_0 B_0 + A_0 \Sigma b_i p^i z^{-i} - B_0 \Sigma a_i p^i z^{-i},$$

and we obtain, for the coefficient of $2\alpha^i \sin i\theta$, since $a_0 = 0$,

$$(a_1 b_{i-1} + a_2 b_{i-2} + \ldots + a_i b_0) + \alpha^2 (-a_1 b_{i+1} + a_{i+1} b_1) + \alpha^4 (-a_2 b_{i+2} + a_{i+2} b_2), \quad \ldots(7)$$

there being no constant term.

2·9. *Fourier expansion of a series expressed in powers of cosines or sines.*

(i) Let

$$C = \Sigma a_i \alpha^i \cos^i \theta = \Sigma b_i (2 \cos\theta)^i, \qquad i = 0, 1, 2, \ldots \quad \ldots(1)$$

Since this is an even function of θ it can be expressed in terms of cosines of multiples of θ. With $z = \alpha \exp.\, \theta\sqrt{-1}$, so that $2\alpha \cos\theta = z + \alpha^2/z$, we have

$$C = \Sigma \frac{a_i}{2^i}\left(z + \frac{\alpha^2}{z}\right)^i = \Sigma \frac{a_i z^i}{2^i}\left(1 + i\frac{\alpha^2}{z^2} + \frac{i(i-1)}{1\,.\,2}\frac{\alpha^4}{z^4} + \ldots + \frac{\alpha^{2i}}{z^{2i}}\right),$$

on expansion by the binomial theorem. As in 2·8 we need to find only the coefficient of z^i, $i > 0$, and to replace z^i by $2\alpha^i \cos i\theta$. The selection gives for the coefficient of $2\alpha^i \cos i\theta$ in C,

$$\frac{a_i}{2^i} + (i+2)\frac{a_{i+2}}{2^{i+2}}\alpha^2 + \frac{(i+4)(i+3)}{1\,.\,2}\frac{a_{i+4}}{2^{i+4}}\alpha^4 + \frac{(i+6)(i+5)(i+4)}{1\,.\,2\,.\,3}\frac{a_{i+6}}{2^{i+6}}\alpha^6 + \ldots \quad (2)$$

The constant term in C is obtained by putting $i=0$ in this formula.

The result shows that the numerical work will be simplified by the use of $b_i = a_i a^i/2^i$. The coefficient of $2\cos i\theta$ is then

$$b_i + (i+2)\,b_{i+2} + \frac{(i+4)(i+3)}{1\,.\,2}\,b_{i+4} + \frac{(i+6)(i+5)(i+4)}{1\,.\,2\,.\,3}\,b_{i+6} + \dots \quad \dots(3)$$

(ii) Let

$$S = \Sigma a_i a^i \sin^i\theta = \Sigma b_i\,(2\sin\theta)^i, \qquad i = 0, 1, 2, \dots.$$

The terms in S with odd values of i will produce sines of odd multiples of θ and those with even values of i, cosines of even multiples of θ. With the same substitution as before, we have

$$S = \Sigma \frac{a_i}{2^i}\left(z - \frac{a^2}{z}\right)^i (-1)^{i/2}$$

$$= \Sigma \frac{a_i z^i}{2^i}\left(1 - i\frac{a^2}{z^2} + \frac{i(i-1)}{1\,.\,2}\frac{a^4}{z^4} - \dots\right)(-1)^{i/2}.$$

By a similar procedure, we obtain, for the coefficient of

$$2\sin(2i+1)\,\theta,$$

$$(-1)^i\left\{b_{2i+1} + (2i+3)\,b_{2i+3} + \frac{(2i+5)(2i+4)}{1\,.\,2}\,b_{2i+5} + \frac{(2i+7)(2i+6)(2i+5)}{1\,.\,2\,.\,3}\,b_{2i+7} + \dots\right\}; \quad \dots\dots(4)$$

and, for the coefficient of $2\cos 2i\theta$,

$$(-1)^i\left\{b_{2i} + (2i+2)\,b_{2i+2} + \frac{(2i+4)(2i+3)}{1\,.\,2}\,b_{2i+4} + \frac{(2i+6)(2i+5)(2i+4)}{1\,.\,2\,.\,3}\,b_{2i+6} + \dots\right\}, \quad \dots\dots(5)$$

the constant term being

$$b_0 + 2b_2 + \frac{4\,.\,3}{1\,.\,2}\,b_4 + \frac{6\,.\,5\,.\,4}{1\,.\,2\,.\,3}\,b_6 + \dots.$$

2·10. *Expansion of a function of a Fourier series.*

(i) It is desired to obtain the expansion of

$$f(a_0 + 2a_1\alpha \cos\theta + 2a_2\alpha^2 \cos 2\theta + \ldots), \quad \ldots\ldots\ldots(1)$$

in the form

$$b_0 + 2b_1\alpha \cos\theta + 2b_2\alpha^2 \cos 2\theta + \ldots,$$

the assumption being made that $f(a_0 + x)$ is expansible in powers of x. Evidently f is an even function of θ.

As before, we put $z = \alpha \exp. \theta\sqrt{-1}$, $\alpha^2 = p$, so that

$$2\alpha^i \cos i\theta = z^i + p^i z^{-i},$$

and recall that it is sufficient to find the coefficients of z^i and then to replace z^i by $2\alpha^i \cos i\theta$ for $i > 0$.

Put $\qquad A = a_1 z + a_2 z^2 + \ldots, \quad B = a_1 \frac{p}{z} + a_2 \frac{p^2}{z^2} + \ldots,$

so that (1) becomes $f(a_0 + A + B)$. This function will be first expanded in powers of p by Taylor's theorem. We obtain

$$f(a_0 + A + B) = f(a_0 + A) + p\left(\frac{\partial f}{\partial p}\right)_0 + \frac{p^2}{2!}\left(\frac{\partial^2 f}{\partial p^2}\right)_0 + \ldots,$$

the suffix denoting that p is put equal to zero.

Put $$f_1 = f(a_0 + A)$$

and denote derivatives of f_1 with respect to a_0 by accents. Then since p is present in f only through B,

$$\left(\frac{\partial f}{\partial p}\right)_0 = \left(\frac{\partial f}{\partial B}\frac{\partial B}{\partial p}\right)_0 = f_1' \frac{a_1}{z},$$

$$\left(\frac{\partial^2 f}{\partial p^2}\right)_0 = \left\{\frac{\partial^2 f}{\partial B^2}\left(\frac{\partial B}{\partial p}\right)^2 + \frac{\partial f}{\partial B}\frac{\partial^2 B}{\partial p^2}\right\}_0 = f_1'' \frac{a_1^2}{z^2} + f_1' \frac{2a_2}{z^2},$$

$$\left(\frac{\partial^3 f}{\partial p^3}\right)_0 = f_1''' \frac{a_1^3}{z^3} + f_1'' \frac{6a_1 a_2}{z^3} + f_1' \frac{6a_3}{z^3}, \text{ etc.}$$

Hence, replacing p by α^2, we obtain

$$f(a_0 + A + B) = f_1 + \frac{\alpha^2}{z} a_1 f_1' + \frac{\alpha^4}{z^2}(a_2 f_1' + \tfrac{1}{2} a_1^2 f_1'')$$

$$+ \frac{\alpha^6}{z^3}(a_3 f_1' + a_1 a_2 f_1'' + \tfrac{1}{6} a_1^3 f_1''') + \ldots\ldots\ldots(2)$$

It will be noticed that all negative powers of z are shown in explicit form in this expression, and that their coefficients are

positive powers of z; thus the lowest power of z required in the second, third, ... coefficients are the first, second, ... We plan, however, to stop at some definite power of α; suppose this power be the seventh. Then since z^i has the factor α^i, we shall need the following expansions:

$$\begin{aligned} &\text{coef. of } \alpha^2/z, &&\text{from } z \text{ to } z^6;\\ &\text{coef. of } \alpha^4/z^2, &&\text{from } z^2 \text{ to } z^5;\\ &\text{coef. of } \alpha^6/z^3, &&\text{from } z^3 \text{ to } z^4;\end{aligned}$$

and these are all that are needed. The work of expansion is thus reduced to operations with positive power series.

We next expand f_1 and its derivatives in powers of A, which contains α as a factor. As far as α^7, the result is, with f_0 set for $f(a_0)$,

$$\begin{aligned} f_1 = f_0 &+ Af_0' + \dots + \frac{A^7}{7!} f_0^{\text{vii}} \\ &+ a_1 \frac{\alpha^2}{z}\left(Af_0'' + \frac{A^2}{2!}f_0''' + \dots + \frac{A^6}{6!}f_0^{\text{vii}}\right) \\ &+ \frac{\alpha^4}{2z^2}\left\{A\,(2a_2 f_0'' + a_1^2 f_0''') + \dots + \frac{A^5}{5!}\,(2a_2 f_0^{\text{vi}} + a_1^2 f_0^{\text{vii}})\right\} \\ &+ \frac{\alpha^6}{6z^3}\left\{A\,(6a_3 f_0'' + 6a_1 a_2 f_0''' + a_1^3 f_0^{\text{iv}}) + \dots \right. \\ &\qquad \left. + \frac{A^4}{4!}\,(6a_3 f_0^{\text{v}} + 6a_1 a_2 f_0^{\text{vi}} + a_1^3 f_0^{\text{vii}})\right\}. \quad \dots\dots(3)\end{aligned}$$

The final step requires the expansions of powers of A in powers of z, the results being required to z^7 in the first line, to z^6 in the second line, and so on. The highest powers are easily formed. We have, in fact,

$$A^7 = a_1^7 z^7, \quad A^6 = a_1^6 z^6 + 6a_1^6 a_2 z^7, \ \dots.$$

The lower powers are conveniently obtained from the binomial theorem by treating $a_1 z + a_2 z^2$ as the first element and the rest of the series as the second element. Thus

$$A^j = (a_1 z + a_2 z^2)^j + j\,(a_1 z + a_2 z^2)^{j-1}(a_3 z^3 + a_4 z^4 + a_5 z^5)$$

will serve for $j \geqq 3$. For $j = 2$, we have

$$\begin{aligned} A^2 = a_1^2 z^2 &+ 2a_1 a_2 z^3 + (a_2^2 + 2a_1 a_3)\, z^4 + (2a_1 a_4 + 2a_2 a_3)\, z^5 \\ &+ (a_3^2 + 2a_1 a_5 + 2a_2 a_4)\, z^6 + (2a_1 a_6 + 2a_2 a_5 + 2a_3 a_4)\, z^7.\end{aligned}$$

After the insertion of these results, the rejection of negative powers of z, and the replacement of z^i by $2\alpha^i \cos i\theta$, we shall have the needed expansion.

An expansion carried out in this manner should, in general, be used only when literal series are desired. If the coefficients have numerical values, the methods of harmonic analysis lead much more easily and directly to the required series. The same remark applies to the function of a sine series, expanded in the next paragraph.

(ii) *The expansion of*

$$f(2a_1 \sin\theta + 2a_2 \sin 2\theta + \ldots).$$

With the same substitutions as before this becomes

$$f(-\sqrt{-1}\,A + \sqrt{-1}\,B),$$

so that the results of (i) are immediately applicable. In general, the resulting series will contain both sines and cosines, but as the important applications are confined to those cases in which f is either an even function of θ, in which case we shall have cosines only, or an odd function of θ, in which we shall have sines only, these two applications alone will be considered.

The expansion (2) can be utilised if in it

(α) a_i be replaced by $a_i\sqrt{-1}$ where it occurs explicitly;

(β) A be replaced by $-A\sqrt{-1}$;

(γ) $f_0' = f_0''' = f_0^{\mathrm{v}} = \ldots = 0$ for f even, $f_0'' = f_0^{\mathrm{iv}} = \ldots = 0$ for f odd.

We thus obtain, after simplifying:

(a) When f is an even function of θ, the expansion is

$$\begin{aligned}
&f_0 - \frac{A^2}{2\,!} f_0'' + \frac{A^4}{4\,!} f_0^{\mathrm{iv}} - \frac{A^6}{6\,!} f_0^{\mathrm{vi}} \\
&+ a_1 \frac{\alpha^2}{z}\left(A f_0'' - \frac{A^3}{3\,!} f_0^{\mathrm{iv}} + \frac{A^5}{5\,!} f_0^{\mathrm{vi}}\right) \\
&+ \frac{\alpha^4}{2z^2}\left(A\, 2a_2 f_0'' + \frac{A^2}{2\,!} a_1^2 f_0^{\mathrm{iv}} - \frac{A^3}{3\,!} 2a_2 f_0^{\mathrm{iv}} - \frac{A^4}{4\,!} a_1^2 f_0^{\mathrm{vi}} + \frac{A^5}{5\,!} 2a_2 f_0^{\mathrm{vi}}\right) \\
&+ \frac{\alpha^6}{6z^3}\left\{A\,(6a_3 f_0'' - a_1^3 f_0^{\mathrm{iv}}) + \frac{A^2}{2\,!} 6a_1 a_2 f_0^{\mathrm{iv}}\right. \\
&\qquad\left. + \frac{A^3}{3\,!}(-6a_3 f_0^{\mathrm{iv}} + a_1^3 f_0^{\mathrm{vi}}) - \frac{A^4}{4\,!} 6a_1 a_2 f_0^{\mathrm{vi}}\right\}. \qquad \ldots(4)
\end{aligned}$$

After the expansion in powers of A as in (i) and the rejection of negative powers of z, we replace z^i by $2a^i \cos i\theta$.

(b) When f is an odd function of θ, the expansion is the following expression divided by $\sqrt{-1}$:

$$
\begin{aligned}
&A f_0' - \frac{A^3}{3!} f_0''' + \frac{A^5}{5!} f_0^{\mathrm{v}} - \frac{A^7}{7!} f_0^{\mathrm{vii}} \\
&\quad + a_1 \frac{a^2}{z} \left(\frac{A^2}{2!} f_0''' - \frac{A^4}{4!} f_0^{\mathrm{v}} + \frac{A^6}{6!} f_0^{\mathrm{vii}} \right) \\
&\quad + \frac{a^4}{2z^2} \left(- A a_1^2 f_0''' + \frac{A^2}{2!} 2a_2 f_0''' + \frac{A^3}{3!} a_1^2 f_0^{\mathrm{v}} \right. \\
&\qquad\qquad\qquad \left. - \frac{A^4}{4!} 2a_2 f_0^{\mathrm{v}} - \frac{A^5}{5!} a_1^2 f_0^{\mathrm{vii}} \right) \\
&\quad + \frac{a^6}{6z^3} \left\{ - A\, 6a_1 a_2 f_0''' + \frac{A^2}{2!} (6a_3 f_0''' - a_1^3 f_0^{\mathrm{v}}) \right. \\
&\qquad\qquad \left. + \frac{A^3}{3!} 6a_1 a_2 f_0^{\mathrm{v}} + \frac{A^4}{4!} (-6a_3 f_0^{\mathrm{v}} + a_1^3 f_0^{\mathrm{vii}}) \right\}. \quad \ldots (5)
\end{aligned}
$$

After following the same processes as before we replace z^i by

$$\sqrt{-1} \,.\, 2a^i \sin i\theta.$$

2·11. *Fourier expansions of* $f(a_0 + 2a_1 \cos\theta)$, $f(a_0 + 2a_1 \sin\theta)$. When $a_2, a_3, \ldots$ are zero, a method dependent on the series

$$Q_i(x) = \frac{1}{i!} \left\{ 1 + \frac{x}{1} \cdot \frac{1}{i+1} + \frac{x^2}{1\,.\,2} \cdot \frac{1}{(i+1)(i+2)} + \ldots \right\}, \quad \ldots (1)$$

which is closely allied to a Bessel function, may be conveniently adopted.

Taylor's theorem may be written

$$f(a_0 + x) = \exp. xD \,.\, f(a_0), \qquad D = \partial/\partial a_0.$$

Hence

$$f(a_0 + a_1 x + a_{-1} x^{-1}) = \exp. a_1 x D \,.\, \exp. a_{-1} x^{-1} D \,.\, f(a_0). \quad \ldots (2)$$

Expand each of the exponentials in powers of x, taking the power $i+k$ in the first and the power k in the second. The product will give the terms containing x^i if we sum for k. It is

$$\sum_k \frac{(a_1 x D)^{i+k}}{(i+k)!} \cdot \frac{(a_{-1} x^{-1} D)^k}{k!} = x^i Q_i (a_1 a_{-1} D^2) \,.\, a_1^i D^i.$$

The similar expression for the coefficient of x^{-i} is evidently found by interchanging $i+k, i$, that is, by interchanging a_1, a_{-1}. Hence

$$f(a_0+a_1x+a_{-1}x^{-1})=\Sigma_i\{(a_1x)^i+(a_{-1}x^{-1})^i\}\\ \times Q_i(a_1a_{-1}D^2)D^i.f(a_0). \quad ...(3)$$

In any application, the operator $D^iQ_i(a_1a_{-1}D^2)$ must be used in the expanded form (1).

For the first Fourier expansion put

$$a_{-1}=a_1, \quad x=\exp\theta\sqrt{-1}, \quad x^i+x^{-i}=2\cos i\theta.$$

The expansion (3) then gives

$$f(a_0+2a_1\cos\theta)=2\Sigma_i a_1{}^i D^iQ_i(a_1{}^2D^2).f(a_0)\cos i\theta, \quad ...(4)$$

where $i=0, 1, 2, \ldots$, the factor 2 being omitted when $i=0$.

For the second Fourier expansion, replace $a_1, -a_1$ by $a_{-1}\sqrt{-1}$, with the same substitute for x. Since

$$x^i-x^{-i}=2\sqrt{-1}\sin i\theta,$$

we get cosines for even values of i and sines for odd values, and

$$f(a_0+2a_1\sin\theta)=2\Sigma_i(-1)^i a_1{}^{2i}D^{2i}Q_{2i}(a_1{}^2D^2).f(a_0)\cos 2i\theta\\ +2\Sigma_i(-1)^i a_1{}^{2i+1}D^{2i+1}Q_{2i+1}(a_1{}^2D^2).f(a_0)\sin(2i+1)\theta, \quad ......(5)$$

with $i=0, 1, 2, \ldots$, the factor 2 in the first series being omitted when $i=0$.

It may be pointed out that

$$x^iQ_i(-x^2)=J_i(2x),$$

where J_i is a Bessel function (2·14), so that the operators may be expressed by these functions. But since the expression in this form involves the presence of imaginaries, it is simpler to use the functions Q_i.

2·12. *Expansion of a power of a Fourier series.* For

$$(a_0+2a_1\alpha\cos\theta+2a_2\alpha^2\cos 2\theta+\ldots)^j$$

we make use of 2·10 (3) with

$$f_0=a_0{}^j, \quad f_0'=ja_0{}^{j-1}, \quad f_0''=j(j-1)a_0{}^{j-2}, \ldots.$$

It is usually found convenient to take out the factor $a_0{}^j$, so that, in the operations, $a_0 = 1$ and $f_0, f_0', \ldots$ are integers when j is an integer.

For
$$(2a_1\alpha \sin\theta + 2a_2\alpha^2 \sin 2\theta + \ldots)^j$$
we apply 2·10 (4) or (5) according as j is even or odd. In these we put
$$f_0{}^{(j)} = j!, \quad f_0{}^{(k)} = 0, \qquad k \neq j,$$
when j is a positive integer.

2·13. *Expansion of the cosine and sine of a Fourier sine series.*

(i) The expansion of
$$\cos(2a_1\alpha \sin\theta + 2a_2\alpha^2 \sin 2\theta + \ldots)$$
is obtained from 2·10 (4) by putting $f_0 = -f_0'' = f_0{}^{\text{iv}} = \ldots = 1$. As far as α^7 this gives
$$\begin{aligned}1 &+ \frac{A^2}{2!} + \frac{A^4}{4!} + \frac{A^6}{6!} - a_1\frac{\alpha^2}{z}\left(A + \frac{A^3}{3!} + \frac{A^5}{5!}\right)\\ &- a_2\frac{\alpha^4}{z^2}\left(A + \frac{A^3}{3!} + \frac{A^5}{5!}\right) + a_1^2\frac{\alpha^4}{2z^2}\left(\frac{A^2}{2!} + \frac{A^4}{4!}\right)\\ &- (a_3 + \tfrac{1}{6}a_1^3)\frac{\alpha^6}{z^3}\left(A + \frac{A^3}{3!}\right) + a_1a_2\frac{\alpha^6}{z^3}\left(\frac{A^2}{2!} + \frac{A^4}{4!}\right). \quad \ldots\ldots(1)\end{aligned}$$
After expansion in powers of A as in 2·10 (i) and the rejection of negative powers of z we replace z^i by $2\alpha^i \cos i\theta$.

(ii) The expansion of
$$\sin(2a_1\alpha \sin\theta + 2a_2\alpha^2 \sin 2\theta + \ldots)$$
is obtained from 2·10 (5) by putting $f_0' = -f_0''' = f_0{}^{\text{v}} = \ldots = 1$. This gives for the function to be divided by $\sqrt{-1}$:
$$\begin{aligned}A &+ \frac{A^3}{3!} + \frac{A^5}{5!} + \frac{A^7}{7!} - a_1\frac{\alpha^2}{z}\left(\frac{A^2}{2!} + \frac{A^4}{4!} + \frac{A^6}{6!}\right)\\ &+ a_1^2\frac{\alpha^4}{2z^2}\left(A + \frac{A^3}{3!} + \frac{A^5}{5!}\right) - a_2\frac{\alpha^4}{z^2}\left(\frac{A^2}{2!} + \frac{A^4}{4!}\right)\\ &+ a_1a_2\frac{\alpha^6}{z^3}\left(A + \frac{A^3}{3!}\right) - (a_3 + a_1^3)\frac{\alpha^6}{z^3}\left(\frac{A^2}{2!} + \frac{A^4}{4!}\right). \quad \ldots\ldots(2)\end{aligned}$$
In the final result, z^i is replaced by $\sqrt{-1}\, 2\alpha^i \sin i\theta$.

2·14. *Bessel's Functions.* The Bessel function of the first kind, $J_j(x)$, may be defined by the series

$$J_j(x) = \frac{1}{j!}\left(\frac{x}{2}\right)^j \left\{1 - \frac{1}{j+1}\cdot\frac{1}{1}\left(\frac{x}{2}\right)^2 + \frac{1}{(j+1)(j+2)}\cdot\frac{1}{1.2}\left(\frac{x}{2}\right)^4 - \dots\right\}, \quad \dots\dots(1)$$

where j is a positive integer. For a negative suffix, we define it by

$$J_{-j}(x) = (-1)^j J_j(x) = J_j(-x). \quad \dots\dots\dots\dots(2)$$

A comparison of coefficients of x^i shows that

$$J_{j-1}(x) + J_{j+1}(x) = \frac{2j}{x} J_j(x), \quad J_{j-1}(x) - J_{j+1}(x) = 2\frac{d}{dx} J_j(x), \quad \dots\dots(3)$$

and that the differential equation

$$\left(\frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx} + 1 - \frac{j^2}{x^2}\right) J_i(x) = 0$$

is satisfied.

The properties most useful for our purposes are deduced from the fact that $J_j(x)$ is the coefficient of z^j in the expansion

$$\exp. \frac{x}{2}\left(z - \frac{1}{z}\right) = \Sigma_j J_j(x) z^j, \qquad j = 0, \pm 1, \dots. \quad \dots\dots(4)$$

This result is shown by expanding $\exp. xz/2$, $\exp.(-x/2z)$ in powers of xz, $-x/z$ respectively and choosing the coefficient of z^j in the product of the two series.

Put $z = \exp. g\sqrt{-1}$ in (4), so that $z - 1/z = 2\sqrt{-1}\sin g$. We obtain

$$\exp.(x \sin g . \sqrt{-1}) = \Sigma_j J_j(x) \exp.(jg\sqrt{-1}), \qquad j = 0, \pm 1, \dots. \quad \dots\dots(5)$$

In this equation, change the sign of g, put $x = ie$ and multiply both members by $\exp.(ig\sqrt{-1})$. We obtain

$$\exp.\{i(g - e\sin g)\sqrt{-1}\} = \Sigma_j J_j(ie) . \exp.\{(i-j) g\sqrt{-1}\}.$$

The real and imaginary parts of this equation give

$$\left.\begin{aligned} \cos i(g - e\sin g) &= \Sigma_j J_j(ie)\cos(i-j)g, \\ \sin i(g - e\sin g) &= \Sigma_j J_j(ie)\sin(i-j)g, \end{aligned}\right\} \qquad j = 0, \pm 1, \dots. \quad \dots\dots(6)$$

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The same process may be applied to (5). With the aid of (2) the results may also be written

$$\left.\begin{aligned}\cos(x\sin g) &= J_0(x) + 2\Sigma_j J_{2j}(x)\cos 2jg,\\ \sin(x\sin g) &= \qquad\qquad 2\Sigma_j J_{2j+1}(x)\sin(2j+1)g,\end{aligned}\right\} \quad j = 1, 2, \ldots. \qquad \ldots\ldots(7)$$

The Bessel function may also be defined by

$$J_j(x) = \frac{1}{\pi}\int_0^\pi \cos(j\phi - x\sin\phi)\,d\phi, \qquad \ldots\ldots\ldots(8)$$

a result which is deducible by the use of the Fourier theorem from (6).

2·15. *The Hypergeometric series.* This series, namely,

$$F(A, B, C, x) = 1 + \frac{A\,.\,B}{1\,.\,C}x + \frac{A(A+1)}{1\,.\,2}\cdot\frac{B(B+1)}{C(C+1)}x^2 + \ldots, \qquad \ldots\ldots(1)$$

includes certain series which are needed in the development of the disturbing function. It satisfies the differential equation,

$$(x^2 - x)\frac{d^2F}{dx^2} + \{x(A + B + 1) - C\}\frac{dF}{dx} + ABF = 0, \quad \ldots(2)$$

and admits of many transformations*, two of which give

$$F(A, B, C, x) = (1 - x)^{-A}\,F\left(A, C - B, C, \frac{-x}{1 - x}\right), \quad \ldots\ldots(3)$$

$$F(A, B, C, x) = (1 - x)^{C-A-B}\,F(C - A, C - B, C, x). \quad \ldots(4)$$

The differential equation may be used to find the expansion of F in powers of y, where $x = a + y$. If this substitution be made in (2) and if we put

$$F = a_0 + a_1 y + a_2 y^2 + \ldots,$$

in the resulting equation, the condition that the coefficient of y^n shall vanish identically is

$$(a^2 - a)(n + 2)\,a_{n+2} + \{n(2a - 1) + a(A + B + 1) - C\}\,a_{n+1} + a_n\frac{(A + n)(B + n)}{n + 1} = 0. \quad \ldots\ldots(5)$$

* See A. R. Forsyth, *Differential Equations*, Chap. VI; Riemann-Weber, *Die part. Diff.-Gleich. der Math. Phys.* vol. 2, pp. 18, 19.

This recurrence formula can be used to find all the coefficients when two of them are known. By direct calculation a_0, a_1 can be obtained from

$$a_0 = F(A, B, C, a), \quad a_1 = \frac{d}{da} F(A, B, C, a),$$

to the required degree of accuracy, and thence a_2, a_3, ... are successively obtained. It may be necessary to carry a_0, a_1 to two or three additional places of decimals in order to compensate for the loss of accuracy which the use of the recurrence formula may produce.

The following formulae are immediately proved:

$$F_C - F_{C+1} = \frac{x}{C}\frac{dF_{C+1}}{dx}, \quad F_A - F_{A+1} = -\frac{x}{A}\frac{dF_A}{dx}, \quad \text{......(6)}$$

$$\frac{dF_{A,B,C}}{dx} = \frac{AB}{C} F_{A+1,B+1,C+1}, \quad \text{...............(7)}$$

where the meaning of the notation is evident. By differentiating these equations and substituting for the derivatives in the differential equation, we can obtain various equations connecting three series for A, $A+1$, $A+2$ with C, B unchanged, or C, $C+1$, $C+2$ with A, B unchanged or with A, B each increased by 1, 2 with C unchanged, etc.

2·16. *Expansion of* $(1-ax)^{-s}(1-a/x)^{-t}$ *in positive and negative powers of* x.

Let us adopt the notation

$$\binom{n}{r}' = \frac{n(n+1)\dots(n+r-1)}{r!}, \quad \binom{n}{0}' = 1, \quad \text{......(1)}$$

the accent being used to avoid confusion with the usual notation

$$\binom{n}{r} = \frac{n(n-1)\dots(n-r+1)}{r!} = (-1)^r\binom{-n}{r}'. \quad \text{......(2)}$$

Expand each of the factors by the binomial theorem. The product is

$$\left\{\Sigma\binom{s}{i}' a^i x^i\right\}\left\{\Sigma\binom{t}{j}' a^j x^{-j}\right\}, \qquad i, j = 0, 1, 2, \dots.$$

The coefficient of x^n, n positive, in this product is obtained by putting $i=n+j$ in the first factor and summing for all values of j. The coefficient is

$$\Sigma_j \binom{s}{n+j}' \binom{t}{j}' \alpha^{2j+n} = \binom{s}{n}' \alpha^n \left\{1 + \frac{s+n}{1+n} \cdot \frac{t}{1} \alpha^2 \right.$$
$$\left. + \frac{(s+n)(s+n+1)}{(1+n)(2+n)} \cdot \frac{t(t+1)}{1 \cdot 2} \alpha^4 + \dots \right\}.$$

The coefficient of x^{-n} is evidently obtained by interchanging t, s.

The most important case is that for which $s=t$: the coefficients of positive and negative powers of x then become equal.

The part of the coefficient within the parentheses may be expressed by the hypergeometric series

$$F(t, s+n, 1+n, \alpha^2),$$

a form which permits of the immediate application of the transformations of 2·15.

When $t=s$, we obtain in this way the following forms for the coefficient of x^n or x^{-n}, namely,

$$\alpha^n \frac{s(s+1)\dots(s+n-1)}{1 \cdot 2 \dots n} f(\alpha^2),$$

where

$$f(\alpha^2) = 1 + \frac{s+n}{1+n} \cdot \frac{s}{1} \alpha^2 + \frac{(s+n)(s+n+1)}{(1+n)(2+n)} \cdot \frac{s(s+1)}{1 \cdot 2} \alpha^4 + \dots \quad \dots\dots(3)$$

$$= \frac{1}{(1-\alpha^2)^s} \left\{1 - \frac{1-s}{1+n} \cdot \frac{s}{1} \frac{\alpha^2}{1-\alpha^2} \right.$$
$$\left. + \frac{(1-s)(2-s)}{(1+n)(2+n)} \cdot \frac{s(s+1)}{1 \cdot 2} \frac{\alpha^4}{(1-\alpha^2)^2} + \dots \right\} \quad \dots\dots(4)$$

$$= \frac{1}{(1-\alpha^2)^{2s-1}} \left\{1 + \frac{1-s}{1+n} \cdot \frac{1+n-s}{1} \alpha^2 \right.$$
$$\left. + \frac{(1-s)(2-s)}{(1+n)(2+n)} \cdot \frac{(1+n-s)(2+n-s)}{1 \cdot 2} \alpha^4 + \dots \right\}. \quad \dots\dots(5)$$

When $n=0$, the coefficient of $f(\alpha^2)$ is 1.

If $x = \exp. \theta\sqrt{-1}$, the coefficients become those of $2\cos n\theta$ and of the constant term in the expansion of $(1 - 2\alpha\cos\theta + \alpha^2)^{-s}$. (Cf. 4·2.)

2·17. *Devices for the numerical calculation of the values of functions defined by power series.* The ratio of two consecutive coefficients of the hypergeometric series and of many other series which occur in celestial mechanics approaches the limit unity, and when the variable is near unity, the calculation may become tedious. The following device is often effective, especially when the coefficients are alternately positive and negative.

The identity

$$a_0 - a_1 x + a_2 x^2 - \ldots = \frac{1}{1+x}\left\{a_0 + \Delta a_0 \frac{x}{1+x} + \Delta^2 a_0 \left(\frac{x}{1+x}\right)^2 + \ldots\right\}, \quad \ldots\ldots(1)$$

where

$$\Delta a_0 = a_0 - a_1,\ \Delta^2 a_0 = a_0 - 2a_1 + a_2,\ \Delta^3 a_0 = a_0 - 3a_1 + 3a_2 - a_3, \ldots, \quad \ldots\ldots(2)$$

is easily proved by expansion of the right-hand member in powers of x. If $a_0, a_1, \ldots$ are positive, the coefficients $\Delta a_0, \Delta^2 a_0, \ldots$ may form a rapidly decreasing series, and the rate of convergence is increased by the fact that $x/(1+x) < x$, when x is positive. For efficient use the transformation should only be started at a term in the given series where the ratio to the succeeding term is less than 2.

A more general form is given by the identity,

$$a_0 - b_1 a_1 x + b_2 a_2 x^2 - \ldots = a_0 f + \Delta a_0 \frac{xf'}{1!} + \Delta^2 a_0 \frac{x^2 f''}{2!} + \ldots, \quad \ldots\ldots(3)$$

where $\Delta a_0, \Delta^2 a_0, \ldots$ are defined as before and

$$f = 1 - b_1 x + b_2 x^2 - \ldots. \quad \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

For efficient use, the series f should be a known function such that the coefficients of $\Delta a_0, \Delta^2 a_0, \ldots$ form a decreasing series This is the case, for example, when

$$f = (1+x)^{-\lambda},$$

where λ is positive and less than unity. The transformation 2·15 (3) from 2·15 (1) may be effected by means of this result.

The ratio of the $(j+2)$th to the $(j+1)$th term of the series 2·16 (4) may be written

$$-\left\{1+\frac{s(1-s)}{j(j+1)}\right\}\frac{j}{j+n+1}\frac{\alpha^2}{1-\alpha^2}.$$

Suppose that we have calculated a few terms of this series, say j of them. The remainder of the series may be written

$$K_j\left(\frac{\alpha^2}{1-\alpha^2}\right)^j\left[1-\left\{1+\frac{s(1-s)}{j(j+1)}\right\}\frac{j}{j+n+1}\frac{\alpha^2}{1-\alpha^2}+\ldots\right]. \quad \ldots\ldots\ldots(5)$$

The formula (3) is applicable if we put

$$a_0=1, \quad a_1=1+\frac{s(1-s)}{j(j+1)}, \ldots$$

$$b_0=1, \quad b_1=\frac{j}{j+n+1}, \quad b_2=\frac{j(j+1)}{(j+n+1)(j+n+2)}, \ldots$$

with $x=\alpha^2/(1-\alpha^2)$.

In the applications s has the values $\frac{1}{2}$, $\frac{3}{2}$, ..., so that if j be suitably chosen a_0, Δa_0, ... form a rapidly decreasing series. The function f defined by (4) with these values of the b_i is the hypergeometric series

$$F(1, j, j+n+1, -x),$$

which satisfies the differential equation,

$$(x^2+x)\frac{d^2F}{dx^2}+\{(2+j)x+j+n+1\}\frac{dF}{dx}+jF=0. \quad \ldots\ldots\ldots(6)$$

A first integral of this equation is

$$(x^2+x)\frac{dF}{dx}+(jx+n+j)F=n+j, \quad \ldots\ldots\ldots\ldots\ldots\ldots(7)$$

since $F=1$ when $x=0$. The final integral is given by

$$\frac{Fx^{n+j}}{(1+x)^n}=(n+j)\int_0^x\frac{x^{n+j-1}}{(1+x)^{n+1}}\,dx.$$

Since n, j are positive integers, the right-hand member can be integrated. When the value of F for any particular value of x has been found from this equation, the first and second derivatives of F can be obtained from (7), (6) and the higher derivatives by successively differentiating (6). When $j=1$, $n=0$ we have $xF=\log(1+x)$.

2·18. *A device for approximating to the derivative of a series.*

Suppose that the sum of the series

$$S=a_0+a_1x+a_2x^2+\ldots \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

is known for a particular value x_0 of x, and that we also know the coefficients up to a_{n-1}. We then know the value of

$$(S-S_n)_0\div x_0{}^n=a_n+a_{n+1}x_0+a_{n+2}x_0{}^2+\ldots. \quad \ldots\ldots\ldots\ldots(2)$$

We have

$$\left(\frac{dS}{dx}-\frac{dS_n}{dx}\right)\div nx^{n-1}=a_n+\frac{n+1}{n}a_{n+1}x+\frac{n+2}{n}a_{n+2}x^2+\dots$$

$$=(S-S_n)\div x^n+\frac{1}{n}(a_{n+1}+2a_{n+2}x+3a_{n+3}x^2+\dots). \quad \dots\dots(3)$$

If we can neglect the last sum in (3) when $x=x_0$, or approximate to it, we obtain a corresponding approximation to $(dS/dx)_0$. We may get a useful approximation because it often happens that an approximate law of relation between the coefficients $a_n, a_{n+1}, \dots$ is known and the divisor n will assist in diminishing the error of the approximation.

Another result of these conditions is an approximation to a_n from (2) better than would be obtained by the approximate law of relation just mentioned.

2·19. *Note on the forms of products of Fourier series with different arguments.*

If we express two Fourier series in the forms

$$\Sigma a_j\,.\,2\cos j\theta, \quad \Sigma a'_{j'}\,.\,2\cos j'\theta', \qquad j, j'=0, 1, 2, \dots,$$

where it is understood that for $j=0, j'=0$, the factor 2 is omitted, then their product may be expressed in the form

$$\Sigma\Sigma a_j a'_{j'}\,.\,2\cos(j\theta\pm j'\theta'),$$

where it is understood that the portions for $j=0, j'=0, j=j'=0$ are

$$\Sigma a'_{j'}\,.\,2\cos j'\theta', \quad \Sigma a_j\,.\,2\cos j\theta, \quad a_0 a_0',$$

that is, no attention is to be paid to the double sign when either j or j' is zero, and the factor 2 is omitted only when $j=j'=0$.

With the same values

$$\Sigma a_j 2\sin j\theta\,.\,\Sigma a'_{j'}2\cos j'\theta' = \quad \Sigma\Sigma a_j a'_{j'}2\sin(j\theta\pm j'\theta'),$$

$$\Sigma a_j 2\sin j\theta\,.\,\Sigma a'_{j'}2\sin j'\theta' = \mp\,\Sigma\Sigma a_j a'_{j'}2\cos(j\theta\pm j'\theta').$$

This method of expression avoids all doubts as to the presence of factors of 2, which may occur if we use the form $\Sigma a_j\cos j\theta$ with a factor $\frac{1}{2}$ when $j=0$. Moreover, it is the natural form which arises when we use exponential methods of expansion.

CHAPTER III

ELLIPTIC MOTION

3·1. The relative motion of two bodies under the Newtonian law of gravitation is a simple dynamical problem which admits a general solution in terms of well-known algebraic and trigonometric functions. Analytically and geometrically the range of the solution is divided into two portions—the elliptic and the hyperbolic—the transition from one to the other giving a special case, the parabolic. It is shown in the elementary text-books that one method of distinction is given by the relations

$$V^2 - 2\mu/r < 0, \ = 0, \ > 0,$$

where V is the relative velocity and r the distance apart at any time; μ is the sum of the masses reckoned in astronomical units. The first case is that of motion in a closed conic section, an ellipse, in which the eccentricity is less than unity; in the second case the conic is a parabola, with the eccentricity equal to unity, and in the third case it is a hyperbola with the eccentricity greater than unity. When the eccentricity is zero, we have circular motion, the limiting case at one end of the range; when it is infinite, the motion is rectilinear, the limiting case at the other end of the range.

In this volume we shall be concerned only with the first case, and the range will be further limited to values of the eccentricity which are small enough for the series, which are developed in powers of this quantity, to be used for numerical calculation during a certain interval of time without too much labour. Expansions in powers of the eccentricity, either implicit or actual, are necessary with the methods developed below, and the greater part of this chapter consists of the formation of those expansions which will be needed in the problem of three bodies.

3·2. *Solution of the Equations for Elliptic Motion.* If we put $R = 0$ in 1·23 (3), (4) and confine the motion to the plane of

reference (see 3·6), these equations take the form

$$\frac{d^2r}{dt^2} - r\left(\frac{dv}{dt}\right)^2 = -\frac{\mu}{r^2}, \quad \frac{d}{dt}\left(r^2\frac{dv}{dt}\right) = 0. \quad \ldots(1), (2)$$

These equations possess the integrals,

$$\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{dv}{dt}\right)^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right), \quad r^2\frac{dv}{dt} = h, \ldots(3), (4)$$

where a, h are arbitrary constants.

The transformation in 1·27 shows that the elimination of t between (1), (3), (4) gives, if $u = 1/r$,

$$\frac{d^2u}{dv^2} + u - \frac{\mu}{h^2} = 0, \quad \left(\frac{du}{dv}\right)^2 + u^2 = \frac{\mu}{h^2}\left(2u - \frac{1}{a}\right).$$
$$\ldots\ldots(5), (6)$$

The solution of the linear equation (5) is

$$u = \frac{\mu}{h^2}\{1 + e\cos(v - \varpi)\}, \quad \ldots\ldots\ldots\ldots\ldots(7)$$

where e, ϖ are arbitrary constants and we suppose that $0 \leqq e < 1$ This must satisfy (6). The substitution of (7) in (6) gives

$$h^2 = \mu a(1 - e^2). \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(8)$$

Hence
$$\frac{1}{r} = \frac{1 + e\cos(v - \varpi)}{a(1 - e^2)}; \quad \ldots\ldots\ldots\ldots\ldots\ldots(9)$$

so that the maximum and minimum of r are $a(1 \pm e)$. Equation (9) is that of an ellipse referred to a focus as origin, the nearer apse of the ellipse having polar coordinates $a(1 - e)$, ϖ.

Let us now introduce the variable X and a constant n defined by

$$r\,dX = n\,dt, \quad \mu = n^2a^3. \quad \ldots\ldots\ldots(10), (11)$$

The investigation of 1·29 with $R = 0$ shows that the equations of motion with X as independent variable are

$$\frac{d^2r}{dX^2} + r - a = 0, \quad \frac{dv}{dX} = \frac{h}{anr} = \frac{a\sqrt{1 - e^2}}{r},$$
$$\ldots\ldots(12), (13)$$

since (8), (11) show that

$$h = \sqrt{\mu a(1 - e^2)} = na^2\sqrt{1 - e^2}. \quad \ldots\ldots\ldots\ldots(14)$$

Since the maximum and minimum of r are $a(1 \pm e)$, the solution of (12) is

$$r = a(1 - e\cos X), \quad \text{..................(15)}$$

where we define X as having the value 0 when $r = a(1-e)$, that is, when $v = \varpi$.

Thence, if ϵ be an arbitrary constant, the solution of (10) gives

$$g \equiv nt + \epsilon - \varpi = X - e\sin X. \quad \text{............(16)}$$

Finally we obtain, from (13),

$$f \equiv v - \varpi = \int_0^X \frac{\sqrt{1-e^2}\,dX}{1 - e\cos X}. \quad \text{............(17)}$$

The integral can be put into any one of the three forms,

$$f = \sin^{-1}\frac{\sqrt{1-e^2}\sin X}{1 - e\cos X} = \cos^{-1}\frac{\cos X - e}{1 - e\cos X}$$

$$= 2\tan^{-1}\left\{\left(\frac{1-e}{1+e}\right)^{\frac{1}{2}}\tan\tfrac{1}{2}X\right\}.$$

We therefore obtain, with the help of (15),

$$r\cos f = a(\cos X - e), \quad r\sin f = a\sqrt{1-e^2}\sin X, \quad \text{......(18), (19)}$$

$$\tan\tfrac{1}{2}f = \left(\frac{1-e}{1+e}\right)^{\frac{1}{2}}\tan\tfrac{1}{2}X. \quad \text{...............(20)}$$

With the definitions (16), (17) of g, f, the following results are easily deduced:

$$\frac{df}{dg} = \frac{a^2\sqrt{1-e^2}}{r^2}, \quad \frac{dr}{dg} = \frac{ae\sin f}{\sqrt{1-e^2}}, \quad \frac{dX}{dg} = \frac{a}{r}. \quad \text{......(21), (22), (23)}$$

The constants $2a$, e are the major axis and eccentricity of the ellipse; ϖ is the longitude of the nearer apse from the initial line, and ϵ, 'the epoch,' is the longitude of the body when it is passing through the nearer apse, so that it defines the origin of the time. The period of revolution is $2\pi/n$; n is called the mean motion. The angles f, X, g are known as the true, eccentric and mean anomalies, respectively, and $nt + \epsilon$ as the mean longitude, usually denoted in this volume by w.

3·3. Frequent use will be made of ϕ, χ, ψ, η defined by

$$\phi = \exp. f\sqrt{-1}, \quad \chi = \exp. X\sqrt{-1}, \quad \psi = \exp. g\sqrt{-1}, \qquad \text{......(1), (2), (3)}$$

$$e(1+\eta^2) = 2\eta, \qquad \text{.....................(4)}$$

so that

$$2\cos if = \phi^i + 1/\phi^i, \quad 2\sqrt{-1}\sin if = \phi^i - 1/\phi^i, \text{ etc.} \qquad \text{......(5)}$$

By writing (4) in the form $\eta = \frac{1}{2}e + \frac{1}{2}e\eta^2$, and applying Lagrange's theorem, (2·2), we can deduce the expansion

$$\eta^j = \left(\frac{e}{2}\right)^j + j\left(\frac{e}{2}\right)^{j+2} + \frac{j}{2\,!}(j+3)\left(\frac{e}{2}\right)^{j+4} + \frac{j}{3\,!}(j+4)(j+5)\left(\frac{e}{2}\right)^{j+6}$$
$$+ \frac{j}{4\,!}(j+5)(j+6)(j+7)\left(\frac{e}{2}\right)^{j+8} + \ldots. \qquad \text{......(6)}$$

The definitions (1), (2), (4) applied to (18), (19) of the previous section give

$$r\phi = \frac{a\chi}{1+\eta^2}\left(1-\frac{\eta}{\chi}\right)^2, \quad r = \frac{a}{1+\eta^2}(1-\eta\chi)\left(1-\frac{\eta}{\chi}\right), \qquad \text{......(7), (8)}$$

$$\phi = \frac{\chi-\eta}{1-\eta\chi}, \qquad \chi = \frac{\phi+\eta}{1+\eta\phi}, \qquad \text{.....................(9)}$$

and, applied to 3·2 (9),

$$\frac{a}{r} = \frac{1+\eta^2}{(1-\eta^2)^2}(1+\eta\phi)\left(1+\frac{\eta}{\phi}\right). \qquad \text{...........(10)}$$

3·4. An important property of these relations when they are expressed as Fourier series is that characteristic of most of the expansions in Chap. II, namely, that the coefficient of $\cos j\theta$ or $\sin j\theta$ is of the form

$$a_0 e^j + a_2 e^{j+2} + a_4 e^{j+4} + \ldots,$$

whether θ be f, X or g. It appeared in the last chapter that if the functions with which we started originally possessed this property, it was retained under the operations to which they were subjected.

It is evident from equations (15), (16), (20) of 3·2 that $r-a$, $g-X$, $f-X$, when expressed in terms of X, have this property. The operations to which they are subjected are those dealt with in the previous chapter, partly expansions in powers and partly changes of variable, and consequently the property is retained. It is apparently not present in such functions as $r^p \cos qf$, $r^p \sin qf$, expressed in terms of g, but it reappears in $r^p \cos q(f-g)$, $r^p \sin q(f-g)$, and the latter can always replace the former in the applications.

Series having this property will be named d'Alembert series. Operations with such series have been treated in Chap. II.

The relations 3·3 (9) show that any function of ϕ in terms of χ, η can be at once transformed into the same function of χ in terms of ϕ, $-\eta$.

3·5. The facts that $1/r$ in terms of f and r in terms of X satisfy linear differential equations of the form

$$\frac{d^2y}{dx^2}+y=\text{const.},$$

furnish the reason for transforming the equations of motion to the forms given in 1·27–1·29. It will be noticed from 3·3 (7) that $r\cos f$, $r\sin f$ in terms of X and $r^{\frac{1}{2}}\cos\frac{1}{2}f$, $r^{\frac{1}{2}}\sin\frac{1}{2}f$ in terms of $\frac{1}{2}X$, have the same property, but no use appears to have been found for this latter pair of expressions.

3·6. *Agreement of results with Kepler's Laws.*

That the motion takes place in a fixed plane is perhaps obvious. It can, however, be proved at once from the equations of 1·23. For if $R=0$, F depends only on r so that $r^2 d\mathrm{v}/dt$, Γ, θ are constant, and $\mathrm{v}=v$.

Kepler's three laws and Newton's deductions from them can be immediately illustrated from the equations of 3·2.

Law II, which states that equal areas are described by the radii of the planets about the sun in equal times, i.e., that the rate of description of areas is constant, means that the right-hand member of 3·2 (2) is zero; it follows that the resultant force is along the radius.

Law I states that the planets move in ellipses with the sun in one focus. If we substitute 3·2 (9), (4) in 1·23 (3), the radial force, $\partial F/\partial r$, will be seen to vary inversely as the square of the distance.

Law III, which states that the squares of the periodic times are proportional to the cubes of the major axes, is not quite exact. The equations $\mu=n^2a^3$, $\tau=2\pi/n$, give

$$a^3/\tau^2=4\pi^2\mu,$$

where μ is the sum of the masses of the sun and planet. The largest planet, Jupiter, has a mass less than 1/1000 that of the sun, so that the difference between Kepler's third law and the exact statement is small. With the observational material used by Kepler, it is not perceptible.

The laws are, in fact, only approximate descriptions of the actual motions: they cease to hold when the mutual attractions of the planets are included. The third law, in particular, becomes a mere definition. We obtain the mean angular velocity of the planet directly from observation and define a certain distance a by means of the equation $\mu = n^2a^3$. The object of this definition is the convenience of calculating by means of equations in which the terms are *obvious* ratios of times and lengths. For example, the equation,

$$\text{acceleration} = \text{mass} \times [\text{length}]^{-2},$$

which is the initial form of a gravitational equation of motion, is transformed to

$$\text{acceleration} = [\text{length}]^3 [\text{time}]^{-2} [\text{length}]^{-2},$$

which is obviously correct in its dimensions.

The constant a so defined is usually called the 'mean distance.' This use of the word 'mean' in the sense of 'average' is incorrect, even when the motion is elliptic. Equation 3·2 (15) shows that it is the average distance if the eccentric anomaly be taken as the independent variable, but it is easily seen from equations 3·2 (9), 3·10 (5) that it is not so when either the true longitude or the time is so used; in the latter case, however, 3·11 (2) shows that $1/a$ is the mean value of $1/r$.

In the actual integration of the equations, it will be seen that the first arbitrary constant to appear is that on which the distance depends: the mean angular velocity is seen later to be a function of this and of other arbitrary constants which have arisen in the integrations. But since, in general, we can deduce the mean angular velocity from observation with much higher relative accuracy than is possible for the mean distance, it is convenient as a final step to adopt it as an arbitrary constant and to express the constant of distance in terms of it and of the other arbitrary constants. When this is done, a is nothing but an abbreviation for $(\mu/n^2)^{\frac{1}{3}}$.

Confusion is often caused by differences in the meanings attached to the letter a. Sometimes it means $(\mu/n^2)^{\frac{1}{3}}$; at other times it signifies a variable or a constant which has this value as a first approximation. The confusion exists throughout the literature, and the only remedy is the discovery of its exact signification wherever it is used.

Similar confusion is often caused by the use of terms in the problem of three bodies, which have a definite meaning in the problem of two bodies. Thus the 'eccentricity' in the former case may be the coefficient of a

certain periodic term in the expression for one of the coordinates; the 'mean anomaly' and 'true anomaly' are used for certain angles; and so on. A qualifying adjective, like 'osculating,' may again alter the meanings. There is usually not much trouble when qualitative descriptions are alone involved; in quantitative work, accuracy of definition is essential.

3·7. *Fourier developments in terms of X.*

The logarithm of 3·3 (9) gives, on account of the definitions of ϕ, χ,

$$f\sqrt{-1} = X\sqrt{-1} + \log(1-\eta/\chi) - \log(1-\eta\chi)$$
$$= X\sqrt{-1} + \eta(\chi - 1/\chi) + \tfrac{1}{2}\eta^2(\chi^2 - 1/\chi^2) + \dots.$$

Hence, by the definition of χ,

$$f = X + 2\Sigma_j \frac{1}{j}\eta^j \sin jX, \qquad j = 1, 2, \dots. \quad \dots\dots(1)$$

Again, from 3·3 (7), we have

$$r^p\phi^q = \left(\frac{a}{1+\eta^2}\right)^p \chi^q(1-\eta\chi)^{p-q}\left(1-\frac{\eta}{\chi}\right)^{p+q}. \quad \dots(2)$$

The binomial theorem gives, for the expansions of the two binomial factors, with the usual notation for the binomial coefficients,

$$\Sigma_k \binom{p-q}{k}(-\eta)^k\chi^k, \quad \binom{p+q}{k}(-\eta)^k\chi^{-k}, \qquad k = 0, 1, 2, \dots.$$

The coefficient of χ^j, with $j = 0, 1, 2, \dots$, in the product is obtained by putting $j+k$ for k in the first sum and summing the product for all the values of k; for that of χ^{-j}, we proceed similarly with the second factor. Thus the product is

$$\Sigma_j(-\eta\chi)^j\,\Sigma_k\binom{p-q}{j+k}\binom{p+q}{k}\eta^{2k} + \Sigma_j\left(\frac{-\eta}{\chi}\right)^j\Sigma_k\binom{p+q}{j+k}\binom{p-q}{k}\eta^{2k},$$

provided the term for $j = 0$, which is the same in both portions, is not repeated. If we write

$$\binom{p-q}{j+k} = \binom{p-q}{j}\frac{p-q-j}{j+1}\cdots\frac{p-q-j-k+1}{j+k},$$

with a similar change in the corresponding binomial coefficient of the second term, and define S_j, S_j' by

$$\left.\begin{aligned}
S_j &= 1 + \frac{p-q-j}{j+1} \cdot \frac{p+q}{1} \eta^2 \\
&\quad + \frac{(p-q-j)(p-q-j-1)}{(j+1)(j+2)} \cdot \frac{(p+q)(p+q-1)}{1\,.\,2} \eta^4 + \ldots, \\
S_j' &= 1 + \frac{p+q-j}{j+1} \cdot \frac{p-q}{1} \eta^2 \\
&\quad + \frac{(p+q-j)(p+q-j-1)}{(j+1)(j+2)} \cdot \frac{(p-q)(p-q-1)}{1\,.\,2} \eta^4 + \ldots,
\end{aligned}\right\} \quad \ldots\ldots(3)$$

the required expansion can be put into the form

$$r^p \phi^q = \left(\frac{a}{1+\eta^2}\right)^p \Sigma_j (-\eta)^j \left\{ \binom{p-q}{j} S_j \chi^{q+j} + \binom{p+q}{j} S_j' \chi^{q-j} \right\}, \quad \ldots\ldots(4)$$

where $j = 0, 1, 2, \ldots$; the term for $j = 0$ being

$$\left(\frac{a}{1+\eta^2}\right)^p \chi^q S_0, \quad S_0 = 1 + \frac{p^2-q^2}{1^2} \eta^2 + \frac{(p^2-q^2)\{p^2-(q-1)^2\}}{1^2\,.\,2^2} \eta^4 + \ldots.$$

By putting for ϕ^q its value $\cos qf + \sqrt{-1} \sin qf$ in terms of f, and similarly for the powers of χ in terms of X, and by equating the real and imaginary parts, we obtain the expansions of

$$r^p \cos qf, \quad r^p \sin qf$$

as Fourier series with argument X.

The formulae for S_j, S_j' are hypergeometric series and are therefore subject to the transformation 2·15 (3). We have, in fact,

$$\begin{aligned}
S_j &= F(-p-q,\ -p+q+j,\ j+1,\ \eta^2) \\
&= (1-\eta^2)^{p+q} F\left(-p-q,\ 1+p-q,\ j+1,\ \frac{-\eta^2}{1-\eta^2}\right).
\end{aligned}$$

Hence, if we put $S_j = (1-\eta^2)^{p+q} T_j$, so that

$$\begin{aligned}
T_j &= 1 + \frac{p+q}{1} \cdot \frac{1+p-q}{j+1} \frac{\eta^2}{1-\eta^2} \\
&\quad + \frac{(p+q)(p+q-1)}{1\,.\,2} \cdot \frac{(1+p-q)(2+p-q)}{(j+1)(j+2)} \left(\frac{\eta^2}{1-\eta^2}\right)^2 + \ldots,
\end{aligned} \quad \ldots\ldots(5)$$

T_j' = value of T_j when the sign of q is changed, and remember that

$$\frac{1-\eta^2}{1+\eta^2}=(1-e^2)^{\frac{1}{2}},$$

we obtain

$$r^p\phi^q=(a\sqrt{1-e^2})^p\,\Sigma_j(-\eta^j)\left\{\binom{p-q}{j}(1-\eta^2)^q T_j\chi^{q+j}\right.$$
$$\left.+\binom{p+q}{j}(1-\eta^2)^{-q}T_j'\chi^{q-j}\right\},\quad \ldots(6)$$

where $j=0, 1, 2, \ldots$, the terms for $j=0$ requiring the factor $\frac{1}{2}$.

In the applications, p takes the values $0, \pm 1, \pm 2, \ldots$, and q the values $0, 1, 2, \ldots$. When $p \gtreqless q$, the series has a finite number of terms. The case $p=q$ is more easily deduced directly from 3·3 (7). It gives

$$r^p\phi^p=\left(\frac{a}{1+\eta^2}\right)^p\chi^p\left(1-\frac{\eta}{\chi}\right)^{2p}$$
$$=\left(\frac{a}{1+\eta^2}\right)^p\Sigma_j\binom{2p}{j}(-\eta)^j\chi^{p-j},\qquad j=0, 1, \ldots$$

from which $r^p\cos p\phi$, $r^p\sin p\phi$ are at once obtained.

The particular case $p=0$, $q=1$, gives

$$\left.\begin{aligned}\cos f&=-\eta+(1-\eta^2)\,\Sigma\eta^{j-1}\cos jX,\\ \sin f&=\phantom{-\eta+{}}(1-\eta^2)\,\Sigma\eta^{j-1}\sin jX,\end{aligned}\right\}\quad j=1, 2, \ldots.\quad \ldots(7)$$

For the case $q=0$, we have $T_j'=T_j$ and the development contains only cosines of multiples of X. We thus obtain from (6)

$$r^p=(a\sqrt{1-e^2})^p\,\Sigma_j(-\eta)^j\binom{p}{j}T_j\,.\,2\cos jX,\quad \ldots\ldots(8)$$

the factor 2 being omitted when $j=0$, and q having the value zero in T_j.

The following are important particular cases with $q=0$. When $p=-1$, $T_j=1$ and the binomial coefficient is $(-1)^j$. Hence

$$\frac{a}{r}=\frac{1}{\sqrt{1-e^2}}(1+2\eta\cos X+2\eta^2\cos 2X+\ldots).\quad \ldots(9)$$

For $p=-2$, we have

$$\binom{-2}{j}=(j+1)(-1)^j,\quad T_j=1+\frac{2}{j+1}\cdot\frac{\eta^2}{1-\eta^2}.$$

Hence

$$\binom{-2}{j} T_j = (-1)^j \left(j + \frac{1+\eta^2}{1-\eta^2}\right) = (-1)^j \left(j + \frac{1}{\sqrt{1-e^2}}\right),$$

and
$$\frac{a^2}{r^2} = (1-e^2)^{-\frac{3}{2}} \Sigma_j \eta^j (1 + j\sqrt{1-e^2}) \, . \, 2\cos jX, \quad \ldots(10)$$

the constant term being $(1-e^2)^{-\frac{3}{2}}$.

3·8. *Fourier developments in terms of f.*

The formula 3·3 (10) gives

$$r^{-p} = \left\{\frac{1+\eta^2}{a(1-\eta^2)^2}\right\}^p (1+\eta\phi)^p \left(1 + \frac{\eta}{\phi}\right)^p.$$

A comparison of the right-hand member with 3·7 (2) when $q = 0$, shows that the development given for r^p in terms of X can be used directly by changing the sign of η, and multiplying by $(1-e^2)^{-p} a^{-2p}$. We obtain

$$\frac{1}{r^p} = \left\{\frac{1}{a\sqrt{1-e^2}}\right\}^p \Sigma \binom{p}{j} T_j \, . \, 2\eta^j \cos jf, \qquad j = 0, 1, \ldots, \ldots(1)$$

where T_j has the value 3·7 (5) with $q = 0$, and the factor 2 is omitted when $j = 0$.

For $p = -2$, we obtain, as in 3·7,

$$r^2 = a^2\sqrt{1-e^2} \, \{1 + 2\Sigma(1 + j\sqrt{1-e^2})\} (-\eta)^j \cos jf, \ldots(2)$$

in which $j = 1, 2, \ldots$.

The series for g in terms of f is found from this last result, by combining it with 3·2 (21) which may be written

$$\frac{dg}{df} = \frac{r^2}{a^2\sqrt{1-e^2}}.$$

An integration with the arbitrary constant so determined that f, g vanish together, gives

$$g = f + 2\Sigma \left\{\frac{1}{j} + \sqrt{1-e^2}\right\} (-\eta)^j \sin jf, \qquad j = 1, 2, \ldots. \quad \ldots(3)$$

It should be noticed that the coefficient of $\sin f$ in this series is

$$-2\eta - 2\eta\sqrt{1-e^2} = -2\eta\left(1 + \frac{1-\eta^2}{1+\eta^2}\right) = -\frac{4\eta}{1+\eta^2} = -2e.$$

The functions of X in terms of f are obtained from those of f in terms of X by interchanging f, X and changing the sign of η, according to a remark in 3·4. They are, from 3·7 (1),

$$X = f + 2\Sigma \frac{1}{j}(-\eta)^j \sin jf, \qquad j = 1, 2, \ldots, \quad \ldots\ldots(4)$$

and from 3·7 (4), with $p = 0$,

$$\chi^q = \Sigma_j \eta^j \left\{ \binom{-q}{j} S_j \phi^{q+j} + \binom{q}{j} S_j' \phi^{q-j} \right\}, \quad \ldots\ldots\ldots(5)$$

where S_j, S_j' have the values 3·7 (3) with $p = 0$, and the value for $j = 0$ requires the factor $\frac{1}{2}$. From these we obtain $\cos qX$, $\sin qX$.

In particular, from 3·7 (7),

$$\left.\begin{aligned} \cos X &= \eta + (1-\eta^2)\,\Sigma\,(-\eta)^{j-1}\cos jf, \\ \sin X &= \phantom{\eta +{}} (1-\eta^2)\,\Sigma\,(-\eta)^{j-1}\sin jf, \end{aligned}\right\} \quad j = 1, 2, \ldots. \quad \ldots(6)$$

Fourier Developments in Terms of the Mean Anomaly

3·9. These developments are deduced from those in terms of X by means of the implicit relation $g = X - e \sin X$. The solution of this equation is avoided by making use of the theorem of 2·5. Applied to the present case, this theorem states that the coefficient of $\cos jg$ in the expansion of $f(X)$ as a Fourier series with argument g, is the same as the constant term in the Fourier expansion of

$$\frac{-2}{j}\sin(jg - je\sin g)\frac{d}{dg}f(g), \quad \ldots\ldots\ldots\ldots(1)$$

and that the constant term in the expansion is the same as that of

$$(1 - e\cos g) f(g). \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

For the coefficient of $\sin jg$, replace the first sin in (1) by '$-$ cos.'

The form of the first factor of (1) shows that developments by means of Bessel functions (2·14) will be needed. In these developments the parameter $\frac{1}{2}e$ is convenient, while in those of functions of f in terms of X, the parameter η was used. Hence in functions of f in terms of g both parameters may appear.

3·10. *Expansions for* $\cos kX$, $\sin kX$, X, r, $r\cos f$, $r\sin f$, *in terms of* g.

When $f(X)=\cos kX$, 3·9 (1) can be written

$$\frac{k}{j}\cos\{(j-k)g-je\sin g\}+\frac{k}{-j}\cos\{(-j-k)g+je\sin g\}.$$

According to 3·14 (6), the constant term in the Fourier expansion of this expression is

$$\frac{k}{j}J_{j-k}(je)+\frac{k}{-j}J_{-j-k}(-je), \quad \text{..............(1)}$$

for $j=1, 2, \ldots$. When $j=0$, equation 3·9 (2) shows that it is zero if $k\neq 1$ and $-\frac{1}{2}e$ if $k=1$.

A similar investigation gives the coefficient of $\sin jg$ in the expansion of $\sin kX$.

Hence, allowing j to receive both positive and negative values, we have, for $k\neq 1$,

$$\begin{matrix}\cos\\ \sin\end{matrix} kX=\Sigma\frac{k}{j}J_{j-k}(je)\begin{matrix}\cos\\ \sin\end{matrix} jg, \qquad j=\pm 1, \pm 2, \ldots \quad \text{...(2)}$$

When $k=1$, it is convenient to make use of the formulae 2·14 (3), although those just written are available if we add $-\frac{1}{2}e$ to the expansion for $\cos kX$. We obtain

$$\left.\begin{aligned}\cos X &= -\tfrac{1}{2}e+2\Sigma\frac{1}{j^2}\frac{d}{de}J_j(je)\cos jg,\\ \sin X &= \qquad\quad 2\Sigma\frac{1}{je}J_j(je)\sin jg,\end{aligned}\right\} \qquad j=1, 2, \ldots \quad \text{...(3)}$$

The expansion for $\sin X$, inserted in the relation

$$X=g+e\sin X,$$

gives
$$X=g+2\Sigma\frac{1}{j}J_j(je)\sin jg, \qquad j=1, 2, \ldots \quad \text{...(4)}$$

The expansions for r, $r\cos f$, $r\sin f$ are obtained from (3) by the use of the relations

$$r=a(1-e\cos X), \quad r\cos f=a\cos X-ae,$$
$$r\sin f=a(1-e^2)^{\frac{1}{2}}\sin X.$$

They give

$$\left.\begin{aligned} r &= a\left(1+\tfrac{1}{2}e^2\right) - 2a\,\Sigma\frac{1}{j^2}e\frac{d}{de}J_j(je)\cos jg,\\ r\cos f &= -\tfrac{3}{2}ae \qquad + 2a\,\Sigma\frac{1}{j^2}\frac{d}{de}J_j(je)\cos jg.\\ r\sin f &= \qquad 2a(1-e^2)^{\frac{1}{2}}\,\Sigma\frac{1}{je}J_j(je)\sin jg, \end{aligned}\right\} \quad j=1,2,\ldots \quad \ldots(5)$$

3·11. *Expansions for a/r, r^2/a^2, a^2/r^2, f, in terms of g.*

For functions which contain a power of r as a factor, it is sometimes better to replace 3·9 (1) by*

$$2\cos(jg - je\sin g)f(g)(1-e\cos g). \quad \ldots\ldots\ldots(1)$$

That the two expressions have the same constant term in the Fourier expansions—all that we need—is evident since we can express their difference as the derivative of a Fourier series.

The expansion of a/r can be obtained from (1) with

$$f(g) = 1/(1-e\cos g),$$

since $r = a(1-e\cos X)$, with the aid of 2·14 (6). It can also be found from 3·10 (4) with the aid of the relation $a/r = dX/dg$. The result is

$$\frac{a}{r} = 1 + 2\Sigma\, J_j(je)\cos jg, \qquad j=1,2,\ldots \quad \ldots\ldots(2)$$

The fact that the constant term of a/r, expressed in terms of g, is unity is an important property.

For the expansion of r^2/a^2, we have

$$\begin{aligned} \frac{r^2}{a^2} &= (1-e\cos X)^2 = 1 + \tfrac{1}{2}e^2 - 2e\cos X + \tfrac{1}{2}e^2\cos 2X \\ &= 1 + \tfrac{3}{2}e^2 + \Sigma\left\{\frac{-2e}{j}J_{j-1}(je) + \frac{e^2}{j}J_{j-2}(je)\right\}\cos jg, \\ &\qquad\qquad j = \pm 1, \pm 2, \ldots, \end{aligned}$$

by 3·10 (2), 3·10 (3). The use of 2·14 (3) enables us to write this

$$\frac{r^2}{a^2} = 1 + \tfrac{3}{2}e^2 - \Sigma\frac{4}{j^2}J_j(je)\cos jg, \qquad j=1,2,\ldots \quad \ldots(3)$$

* The functional $f(g)$ has, of course, no relation to the true anomaly f.

The relation between the expansions (2), (3) is given by

$$\frac{d^2}{dt^2}r^2 = 2\left(\frac{\mu}{r} - \frac{\mu}{a}\right),$$

with $\mu = n^2a^3$. This equation is easily deduced from 3·2 (1) and 3·2 (3).

For the expansion of a^2/r^2, we make use of 3·11 (1) with $f(g) = (1 - e\cos g)^{-2}$, so that the coefficient of $\cos jg$ is the constant term in the Fourier expansion of

$$2\cos(jg - je\sin g)\,.\,(1 - e\cos g)^{-1}. \qquad \text{.........(4)}$$

The expansion of $1/(1 - e\cos X)$ in terms of X, e is given by 3·7 (9). Hence that of the second factor in terms of g, η, is

$$(1 - e^2)^{-\frac{1}{2}}(1 + 2\Sigma\eta^i\cos ig), \qquad i = 1, 2, \ldots,$$

or

$$(1 - e^2)^{-\frac{1}{2}}\,\Sigma\eta^{|i|}\cos ig, \qquad i = 0, \pm 1, \pm 2, \ldots.$$

With the use of this result, (4) may be written

$$2(1 - e^2)^{-\frac{1}{2}}\,\Sigma\,\eta^{|i|}\cos\{(i + j)g - je\sin g\},$$

and, by 2·14 (6), the constant term in the Fourier expansion of this function is

$$2(1 - e^2)^{-\frac{1}{2}}\,\Sigma\,\eta^{|i|}J_{i+j}(je), \qquad i = 0, \pm 1, \pm 2, \ldots.$$

This is the coefficient of $\cos jg$ in the expansion of a^2/r^2.

The application of 3·9 (2) shows that the constant term in the expansion of a^2/r^2 is $(1 - e^2)^{-\frac{1}{2}}$, and 2·14 (1) shows that $J_i(0) = 0$ except for $i = 0$ when it is unity. The change of i into $i + j$ in the previous expression for the coefficient of $\cos jg$ therefore gives

$$\frac{a^2}{r^2} = 2(1 - e^2)^{-\frac{1}{2}}\,\Sigma_j\Sigma_i\eta^{|i-j|}J_i(je)\cos jg, \qquad \left.\begin{matrix} i = 0, \pm 1, \ldots, \\ j = 0, \quad 1, \ldots, \end{matrix}\right\} \ldots(5)$$

the factor 2 being omitted for the value $j = 0$.

The expansion for f is deduced by inserting this result in 3·2 (21), namely, in

$$\frac{df}{dg} = \frac{a^2(1 - e^2)^{\frac{1}{2}}}{r^2},$$

and integrating with the condition that f, g are to vanish together.

The result is

$$f = g + \Sigma_j \frac{2}{j} \Sigma_i \eta^{|i-j|} J_i(je) \sin jg, \quad \text{...........(6)}$$

where $\quad i = 0, \pm 1, \pm 2, \ldots; \quad j = 1, 2, \ldots.$

While this formula is quite general, it is not very convenient for the actual calculation of any coefficient in powers of e or η, partly because either of these parameters must be expressed in terms of the other, and partly because there are $j+1$ terms of the same order in the coefficient of $\sin jg$. The term of lowest order in any coefficient is, however, easily found, since for this term we can put

$$\eta = \tfrac{1}{2} e, \quad J_i(je) = (je/2)^i \div i!, \qquad i = 0, 1, \ldots j$$

Hence the principal term in the coefficient of $\sin jg$ is

$$\frac{2}{j}\left(\frac{e}{2}\right)^i \left(1 + \frac{j}{1!} + \frac{j^2}{2!} + \ldots + \frac{j^j}{j!}\right). \quad \text{...........(7)}$$

The portions depending on higher powers of e in this coefficient will not be developed in detail. If we adopt the definition

$$j_k = \frac{j^{j-k}}{(j-k)!}, \quad \text{.......................(8)}$$

the coefficient of $(e/2)^{j+2}$ will be found to be

$$\frac{2}{j}\{1 \cdot j_1 + 2 \cdot j_2 + \ldots + (j-1) j_{j-1}\} - 2(j_1 + j_2 + \ldots + j_{j-1}) - \frac{4j^{j+1}}{(j+1)!} \quad \text{......(9)}$$

and that of $(e/2)^{j+4}$,

$$\begin{aligned} \frac{1}{j}\{1 \cdot 4 \cdot j_1 &+ 2 \cdot 5 \cdot j_2 + \ldots + (j-1)(j+2) j_{j-1}\} \\ &- 2(2 \cdot j_1 + 3 \cdot j_2 + \ldots + j \cdot j_{j-1}) \\ &+ j^2\left(\frac{j_1}{j} + \frac{j_2}{j-1} + \ldots + \frac{j_{j-1}}{2}\right) \\ &+ j^2 + 1 + \frac{2j^{j+1}}{(j+2)!}(j^2 - j - 1). \quad \text{..............(10)} \end{aligned}$$

3·12. *Expansion of any power of r by recurrence.*

The expansions of other special functions of r, f may be obtained from the equations of motion. Thus, if we put $x=r\cos f$, $y=r\sin f$, the equations satisfied by x, y are

$$\frac{d^2x}{dt^2}=-\frac{\mu x}{r^3},\quad \frac{d^2y}{dt^2}=-\frac{\mu y}{r^3}.$$

As we have already found the expansions of x, y, these equations give the expansions of $\cos f/r^2$, $\sin f/r^2$.

Again the equation

$$\frac{d^2r}{dt^2}-\frac{h^2}{r^3}=-\frac{\mu}{r^2},\quad h^2=\mu a(1-e^2),$$

gives the expansion of $1/r^3$ when those of r, $1/r^2$ are known.

In general, the equation

$$\frac{1}{p+2}\cdot\frac{d^2}{dt^2}r^{p+2}=(2p+1)\mu r^{p-1}-(p+1)\frac{\mu}{a}r^p-ph^2r^{p-2},$$

which is deducible from (1), (3), (4) of 3·2, can be used to obtain, by recurrence, the series for r^p for all values of p, when those for certain values have been obtained.

3·13. *Expansions of* $r^p\cos qf$, $r^p\sin qf$.

These expansions for values of p, q other than those just considered can be dealt with by first expanding in multiples of X by 3·7 (6), and then using 3·10 (2) to transform to multiples of g. The series suffer from the defect of that for f mentioned in 3·11, namely, that there are $j+1$ terms of the same order in the coefficients of $\cos jg$, $\sin jg$.

If, however, we do not need such expansions beyond e^7, the extensive tables given by Cayley* for various values of p, q and for other functions are available and will serve for most purposes. In cases where this degree of accuracy is not sufficient, numerical values are usually used and then the method of numerical harmonic analysis (3·17) is available.

A combination of the results obtained from the literal and numerical developments by the method indicated at the end of 2·18 will give an approximation to the terms of order e^8 in certain of the coefficients. The method developed in that article also increases the accuracy of derivatives with respect to e, when they are needed.

* *Mem. R.A.S.* vol. 29, pp. 191–306; *Coll. Papers*, vol. 3.

3·14. *The constant term in the expansion of $r^p \cos qf$ in terms of g.*

By means of the relation 3·2 (9), this function is immediately expressed as a function of f. According to the theorem of 2·5, the constant term, when it is expressed as a function of g, is the same as the constant term of

$$r^p \cos qf \,.\, \frac{dg}{df},$$

expressed as a function of f. Hence, by 3·2 (21), we need the constant term of

$$r^{p+2} \cos qf \div a^2 (1-e^2)^{\frac{1}{2}},$$

expressed as a function of f.

The expansion of r^{p+2} as a function of f is obtained from 3·8 (1) by putting $-p-2$ for p. We thus need the constant term of

$$a^p (1-e^2)^{\frac{1}{2}p+\frac{1}{2}} \Sigma \binom{-p-2}{j} T_j \,.\, 2\eta^j \cos jf \cos qf,$$

where in T_j we put $-p-2$ for p, $q=0$. There is only one constant term in this and it is evidently given by $j=q$, that is, it is

$$a^p (1-e^2)^{\frac{1}{2}p+\frac{1}{2}} \binom{-p-2}{q} T_q \eta^q, \qquad\qquad (1)$$

where

$$\begin{aligned} T_q &= 1 + \frac{-p-2}{1} \cdot \frac{-p-1}{q+1} \frac{\eta^2}{1-\eta^2} \\ &\quad + \frac{(-p-2)(-p-3)}{1 \,.\, 2} \cdot \frac{(-p-1)(-p)}{(q+1)(q+2)} \left(\frac{\eta^2}{1-\eta^2}\right)^2 + \dots \\ &= 1 + \frac{(p+1)(p+2)}{1 \,.\, (q+1)} \frac{\eta^2}{1-\eta^2} \\ &\quad + \frac{p(p+1)(p+2)(p+3)}{1 \,.\, 2 \,.\, (q+1)(q+2)} \left(\frac{\eta^2}{1-\eta^2}\right)^2 + \dots . \end{aligned}$$

It is evident that T_q is a finite series for all negative integral values of p; it becomes unity for $p = -1, -2$.

3·15. *The expansion of r^p in terms of g.*

The constant term in this expansion is obtained by putting $q=0$ in 3·14 (1). The coefficient of $(e/2)^j \cos jg$ is found to be, with the help of the notation 3·11 (8),

$$-\frac{2}{j}\left\{1 \binom{p}{1} j_1 - 2 \binom{p}{2} j_2 + 3 \binom{p}{3} j_3 - \dots\right\}; \qquad (1)$$

and that of $(e/2)^{j+2}\cos jg$,

$$\frac{2p}{j+1}j_0+2\left\{1\binom{p}{1}j_0-2\binom{p}{2}j_1+3\binom{p}{3}j_2-\ldots\right\}$$
$$-\frac{2}{j}\left\{1.3\binom{p}{3}j_1-2.4\binom{p}{4}j_2+3.5\binom{p}{5}j_3-\ldots\right\}. \quad \ldots(2)$$

The series in each brace stops at the suffix j.

3·16. *Literal developments to the seventh order.*

The following detailed developments may be found useful for reference. The notation

$$\mathfrak{e}=\tfrac{1}{2}e,$$

is used.

$$g=f-4\mathfrak{e}\sin f+(3\mathfrak{e}^2+2\mathfrak{e}^4+3\mathfrak{e}^6)\sin 2f-\left(\frac{8}{3}\mathfrak{e}^3+4\mathfrak{e}^5+8\mathfrak{e}^7\right)\sin 3f$$
$$+\left(\frac{5}{2}\mathfrak{e}^4+6\mathfrak{e}^6\right)\sin 4f-\left(\frac{12}{5}\mathfrak{e}^5+8\mathfrak{e}^7\right)\sin 5f$$
$$+\frac{7}{3}\mathfrak{e}^6\sin 6f-\frac{16}{7}\mathfrak{e}^7\sin 7f. \quad \ldots(1)$$

$$f=g+\left(4\mathfrak{e}-2\mathfrak{e}^3+\frac{5}{3}\mathfrak{e}^5+\frac{107}{36}\mathfrak{e}^7\right)\sin g+\left(5\mathfrak{e}^2-\frac{22}{3}\mathfrak{e}^4+\frac{17}{3}\mathfrak{e}^6\right)\sin 2g$$
$$+\left(\frac{26}{3}\mathfrak{e}^3-\frac{43}{2}\mathfrak{e}^5+\frac{95}{4}\mathfrak{e}^7\right)\sin 3g+\left(\frac{103}{6}\mathfrak{e}^4-\frac{902}{15}\mathfrak{e}^6\right)\sin 4g$$
$$+\left(\frac{1097}{30}\mathfrak{e}^5-\frac{5957}{36}\mathfrak{e}^7\right)\sin 5g+\frac{1223}{15}\mathfrak{e}^6\sin 6g$$
$$+\frac{47273}{252}\mathfrak{e}^7\sin 7g. \quad \ldots\ldots(2)$$

$$\frac{r}{a}=1+2\mathfrak{e}^2-\left(2\mathfrak{e}-3\mathfrak{e}^3+\frac{5}{6}\mathfrak{e}^5-\frac{7}{72}\mathfrak{e}^7\right)\cos g$$
$$-\left(2\mathfrak{e}^2-\frac{16}{3}\mathfrak{e}^4+4\mathfrak{e}^6\right)\cos 2g-\left(3\mathfrak{e}^3-\frac{45}{4}\mathfrak{e}^5+\frac{567}{40}\mathfrak{e}^7\right)\cos 3g$$
$$-\left(\frac{16}{3}\mathfrak{e}^4-\frac{128}{5}\mathfrak{e}^6\right)\cos 4g-\left(\frac{125}{12}\mathfrak{e}^5-\frac{4375}{72}\mathfrak{e}^7\right)\cos 5g$$
$$-\frac{108}{5}\mathfrak{e}^6\cos 6g-\frac{16807}{360}\mathfrak{e}^7\cos 7g. \quad \ldots\ldots(3)$$

$$\frac{a}{r} = 1 + \left(2e - e^3 + \frac{1}{6}e^5 - \frac{1}{72}e^7\right)\cos g$$
$$+ \left(4e^2 - \frac{16}{3}e^4 + \frac{8}{3}e^6\right)\cos 2g + \left(9e^3 - \frac{81}{4}e^5 + \frac{729}{40}e^7\right)\cos 3g$$
$$+ \left(\frac{64}{3}e^4 - \frac{1024}{15}e^6\right)\cos 4g + \left(\frac{625}{12}e^5 - \frac{15625}{72}e^7\right)\cos 5g$$
$$+ \frac{648}{5}e^6 \cos 6g + \frac{117649}{360}e^7 \cos 7g. \quad\ldots\ldots(4)$$

It is evident that the expressions for f, r are d'Alembert series with respect to the association of powers of e with multiples of g or ϖ (3·4).

3·17. *Numerical developments by harmonic analysis.*

When the numerical value of e is given, the most rapid and accurate method for computing the functions is that of numerical harmonic analysis (App. A). This method requires the calculation of the functions for a few special values of the independent variable. The calculation presents no difficulties when either the eccentric or the true anomaly is taken as the independent variable; the formulae in 3·2 are available for the purpose.

When the independent variable is g, the first step is the solution of the equation $g = X - e \sin X$, for each special value of g. For a low degree of accuracy, tables for the purpose are available*: methods for the correction of these values are given. For high accuracy, the method given below will be found convenient.

When the special values of X have been obtained, those of r, f and thence of any functions of r, f are found from

$$r = a(1 - e \cos X),$$

with any one of the formulae 3·2 (18), (19), (20).

The considerable increase in accuracy obtained with the use of numerical harmonic analysis is due to the fact that in most of the series with which we have to deal, the rate of convergence along the coefficients A_j, in the series $\Sigma A_j \alpha^j \cos jg$ or $\Sigma A_j \alpha^j \sin jg$, is more rapid than that of A_j expressed

* See, for example, those of Boquet, *Obs. d'Abbadia*, Hendaye, and of J. Bauschinger, *Tafeln zur Theor. Astr.* Leipzig.

as a series in powers of α^2, especially for large values of j, unless α is very small. A detailed examination of the errors produced in any coefficient by the neglect of the higher terms with any given set of special values of g will show how this result is obtained*.

Numerical Solution of Kepler's equation. When high accuracy is required, it may be obtained rapidly by a formula obtained as follows.

Put $X = g + x$, so that Kepler's equation may be written

$$x = e \sin (g + x). \qquad \text{.............................(1)}$$

Hence

$$\begin{aligned} \sin x &= x - \tfrac{1}{6}x^3 + \tfrac{1}{120}x^5 - \dots \\ &= e \sin (g+x) - \tfrac{1}{6}e^3 \sin^3 (g+x) + \tfrac{1}{120} e^5 \sin^5 (g+x) - \dots \qquad \text{...(2)} \end{aligned}$$

Calculate C, x_0 from

$$C \cos x_0 = 1 - e \cos g, \quad C \sin x_0 = e \sin g. \qquad \text{...............(3)}$$

These give

$$C^2 = 1 + e^2 - 2e \cos g, \quad C \sin (x_0 + g) = \sin g, \quad e \sin (x_0 + g) = \sin x_0. \qquad \text{......(4)}$$

With the aid of (3), equation (2) may be written

$$C \sin (x - x_0) = -\tfrac{1}{6} e^3 \sin^3 (g+x) + \tfrac{1}{120} e^5 \sin^5 (g+x) - \dots \qquad \text{...(5)}$$

If e^3 be neglected, we have $x = x_0$, the error being of order $e^3/6$. If we put $x = x_0$ in the right-hand member of (5), the maximum error of its first term is found to be of order $e^6/46$. Hence the formula

$$C \sin (x - x_0) = -\tfrac{1}{6} e^3 \sin^3 (g + x_0) \qquad \text{....................(6)}$$

gives x with an error of order $e^6/46$ or $e^5/120$.

By the use of (4) alternative forms for calculation are seen to be

$$\sin (x - x_0) = -\frac{1}{6C} \sin^3 x_0 = -\frac{e^3}{6C^4} \sin^3 g,$$

these giving the same results as (6). Should still higher accuracy be needed, it can be obtained by substituting the value of x thus obtained in the right-hand member of (5), but this will very rarely be necessary. For $e < {\cdot}14$, the error of X found from (6) is less than $0''{\cdot}1$.

* An example will be found in *Mon. Not. R.A.S.* vol. 88, p. 631.

CHAPTER IV

THE DEVELOPMENT OF THE DISTURBING FUNCTION

4·1. In this chapter are given methods for expressing the disturbing function as a sum of periodic terms when for the coordinates are substituted their expressions in terms of the elliptic elements given in Chap. III.

The disturbing function for planetary action obtained in 1·10 is

$$R = \frac{m'}{\Delta} - \frac{m' r \cos S}{r'^2}, \qquad\text{.....................(1)}$$

where r, r' are the distances of the two planets from the sun, Δ is the distance between them, S is the angle between the radii r, r', and m' is the mass of the disturbing planet. Hence

$$\Delta^2 = r^2 + r'^2 - 2rr' \cos S. \qquad\text{................(2)}$$

We have seen also in 1·10, that if the plane of motion of m' be taken as the plane of reference,

$$\left.\begin{aligned}\cos S &= \cos(v-\theta)\cos(v'-\theta) + \cos I \sin(v-\theta)\sin(v'-\theta)\\ &= \cos^2 \tfrac{1}{2} I \cos(v-v') + \sin^2 \tfrac{1}{2} I \cos(v+v'-2\theta).\end{aligned}\right\} \qquad\text{......(3)}$$

In this formula, I is the angle between the two orbital planes, θ is the longitude of the node of the orbital plane of the disturbed planet from a fixed line in the plane of reference, v' is the longitude of the disturbing planet from the same fixed line, and v is that of the disturbed planet reckoned to the node and then along its orbital plane to the body.

If ϖ', ϖ be the longitudes of the nearer apses reckoned in the same manner as v', v, respectively, and if f', f be the true anomalies, we have

$$v' = f' + \varpi', \quad v = f + \varpi. \qquad\text{..................(4)}$$

The substitution of (4), (3) and (2) in (1) gives R as a function of r, r', f, f', ϖ, ϖ', I, θ. The results of Chap. III show how r, r', f, f' may be expressed as functions of the true, eccentric

or mean anomalies. There is thus no difficulty in expressing R as a function of these angles; the problem is the expansion of R into a sum of sines or cosines whose arguments are multiples of them.

The changes necessary when the plane of reference is arbitrary are given in 1·32.

4·2. Suppose that we put the eccentricities e, e' and the inclination I equal to zero. Then the true anomalies f, f', the eccentric anomalies X, X', and the mean anomalies g, g' are respectively equal and r, r' reduce to a, a'. The disturbing function becomes

$$R = \frac{m'}{(a^2 + a'^2 - 2aa' \cos S)^{\frac{1}{2}}} - \frac{m' a \cos S}{a'^2},$$

with $$S = v - v' = g + \varpi - g' - \varpi'.$$

The first term can be expressed as a cosine Fourier series with argument S; the second term is already in the required form.

Suppose $a < a'$ and put $a/a' = \alpha$. Then

$$R = \frac{m'}{a'}(1 - 2\alpha \cos S + \alpha^2)^{-\frac{1}{2}} - \frac{m'}{a'}\alpha \cos S$$

$$= \frac{m'}{a'}\{1 - \tfrac{1}{2}\alpha^2 + \tfrac{3}{8}(2\alpha \cos S - \alpha^2)^2 + \ldots\},$$

on expansion by the binomial theorem. The various powers of $\cos S$ can be replaced by cosines of multiples of S which will then have coefficients expanded in powers of α^2; the general form of the expansion is given in 2·16.

The practical difficulties in connection with this expansion are due to the need for using values of α which are frequently as large as ·7 and to the fact that the coefficients may be needed to five or more significant figures. If the literal series were used, some dozens of terms in a coefficient would often be needed and the work thus become extremely laborious: not infrequently also, some eight or ten multiples of S are required. Thus one problem is the construction of a set of devices for the rapid calculation of these coefficients.

The disappearance of the term $\alpha \cos S$ from the expansion has important consequences in satellite theory where α is very small. In the planetary theory it simply has the effect of diminishing to some extent the terms with argument S, so that those with arguments S, $2S$ have coefficients of about the same order of magnitude in the coordinates.

4·3. When the eccentricities and inclination are not zero, the only available methods for development depend on expansions, implicit or explicit, in powers of these parameters. As far as their magnitudes are concerned, the problem is less difficult than with α, because they usually have values in the neighbourhood of ·1. In exceptional cases, one or two of them may rise to ·4 or ·5: beyond this limit, the expansions are useless for numerical calculation and, in general, the results will have doubtful accuracy for values greater than ·3.

A much more far-reaching effect is produced by the introduction of multiples of the anomalies, other than those of their difference. When the disturbing function is expressed in terms of the time, these multiples take the form $jg \pm j'g'$, where j, j' are positive integers, and the coefficient of the term which has this angle as argument contains the power $|j \pm j'|$ of the eccentricities or inclination. The coefficient of t in the angle is $jn \pm j'n'$, and when an integration is performed this quantity will appear as a divisor. The divisors with the upper sign will tend to diminish the coefficient, but those with the lower sign may increase it.

Consider the expression

$$\frac{j}{j'} - \frac{n'}{n}. \qquad \text{.........................(1)}$$

Since n, n' are observed quantities, we can always find integers which will render this expression as small as we wish, so that integrals involve discontinuities which may require special treatment. It has been pointed out, however, that a term with argument $jg - j'g'$ contains as a factor of its coefficient the power $|j - j'|$ of the eccentricities and inclination, so that for large values of j, j', the factor is very small. From the point of view of the applications, the cases of interest are those in which j/j'

has such values as $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{5}$, ..., and in which the expression (1) is small.

Since the coordinates of the planet will contain the integrals of such terms, the relative accuracy of the results will be diminished unless the corresponding coefficients be taken to more places of decimals. It is this requirement which constitutes the central difficulty in the development of the disturbing function: a few coefficients of a given order with respect to the eccentricities and inclination are needed to a higher degree of accuracy than the remainder of the terms of the same order. The problem is practical rather than mathematical, namely, the avoidance of extensive calculations of numerous terms, only a few of which are ultimately retained.

4·4. In the majority of the older methods, the time is used as the independent variable, requiring the disturbing function to be expressed in terms of the *mean* anomalies. There are several methods of approach. One is to express it first in terms of the true anomalies by means of the equations,

$$r = \frac{a(1-e^2)}{1+e\cos f}, \quad r' = \frac{a'(1-e'^2)}{1+e'\cos f'}, \qquad \text{.........(1)}$$

and, after expansion, to proceed to its expression in terms of the mean anomalies by means of the relations developed in 3·10–3·16.

A second method is the expression of the coordinates in terms of the eccentric anomalies, by means of the relations,

$$\left.\begin{aligned} r &= a(1-e\cos X), \\ r\cos f &= a(\cos X - e), \\ r\sin f &= a\sqrt{1-e^2}\sin X, \end{aligned}\right\} \qquad \text{...............(2)}$$

with similar expressions for the disturbing planet, and then to express the results in terms of the mean anomalies through the use of the implicit relations

$$g = X - e\sin X, \quad g' = X' - e'\sin X'. \qquad \text{.........(3)}$$

Still another method is to proceed straight to the mean anomalies from r, f, r', f' by means of the series developed in the later sections of Chap. III.

If an independent variable other than the time is used, t is eliminated through the relations

$$g = nt + \epsilon - \varpi, \quad g' = n't + \epsilon' - \varpi', \qquad \text{.............(4)}$$

so that the coordinates may be expressed in terms of the variable chosen.

4·5. A further distinction between the methods arises according as a literal development in powers of e, e', I is made, or as numerical values are substituted for the elements of the ellipses from the outset. When the time is the independent variable, the expansion contains multiples of four angles: g, g' and the differences of ϖ, ϖ', θ. If the numerical values be used for the latter, as well as for e, e', I, the disturbing function can be expressed in the form

$$\frac{m'}{a'} A_{j,j'} \cos(jg + j'g') + \frac{m'}{a'} B_{j,j'} \sin(jg + j'g'),$$

where j, j' receive positive and negative integral values, and the A, B are numerical coefficients. The abbreviation of the work is evidently very great. On the other hand, in a numerical method it is difficult to find a few coefficients of high order to more places of decimals without taking the whole, or the greater part of the work to the same degree of accuracy. A further loss with the numerical method is due to the fact that the derivatives of R with respect to the elements, or to some of them, are needed, and these require the calculation of at least three functions when numerical values are used from the outset.

A definite set of rules to fit all cases should be avoided if much unnecessary calculation is not to be carried out. Each case should be examined in some detail, especially the calculations needed for the long-period terms, that is, those for which $jn - j'n'$ is small, and that plan adopted which would seem to give the results needed most efficiently for the case in hand. Familiarity with one method is to some extent time-saving but the gain does not usually balance the loss when a choice of methods is available and advantage is taken of the choice.

4·6. *General methods for expansion in powers of the eccentricities.*

The methods adopted here involve the use of the theorem,

$$F(px) = p^D F(x), \quad D = x\frac{d}{dx}, \qquad \text{...........(1)}$$

proved in 2·7, together with a variation of this theorem found by putting

$$p = \exp. E\sqrt{-1}, \quad x = \exp. \psi,$$

so that
$$x\frac{d}{dx} = \frac{1}{\sqrt{-1}}\frac{d}{d\psi}, \quad p^D = \exp. E\frac{d}{d\psi},$$

and

$$F(\exp. E\sqrt{-1} . \exp. p\psi\sqrt{-1}) = \exp. E\frac{d}{d\psi} F(\exp. \psi\sqrt{-1}). \qquad \text{......(2)}$$

The two formulae take care of all developments along powers of the eccentricities, the former, in general, for linear coordinates, and the latter for angular coordinates.

Put

$$r_0 = a \text{ . function of } e, \quad r_0' = a' \text{ . function of } e', \quad r_0/r_0' = \alpha,$$

where the functions of e, e' are at our disposal but reduce to unity when $e = 0$, $e' = 0$, and let

$$r = r_0\rho, \quad r' = r_0'\rho', \qquad \text{....................(3)}$$

so that ρ, ρ' also become unity when e, e' vanish. Then since R is a homogeneous function of r, r' of degree -1, and since we assume that $\alpha < 1$, we may write it

$$\frac{m}{r'}F\left(\frac{r}{r'}\right) = \frac{m}{r_0'\rho'}F\left(\frac{\rho}{\rho'}\alpha\right) = \frac{m}{r_0'}\rho^D\rho'^{-D-1}F(\alpha), \quad D = \alpha\frac{d}{d\alpha}, \qquad \text{......(4)}$$

by a double application of (1). The eccentricities, so far as they occur through r, r', are contained explicitly in the factors outside the functional sign only; whether they are present in α or not is immaterial to the developments of this chapter.

Again, regarding R as a function of f, f', and putting

$$f = \psi + E, \quad f' = \psi' + E', \qquad \text{..............(5)}$$

where E, E' vanish with e, e' and ψ, ψ' are independent of e, e', and remembering that f, f' occur only under the signs sine and cosine, we have, by (2),

$$F(\exp. f\sqrt{-1}) = F(\exp. E\sqrt{-1} . \exp. \psi\sqrt{-1})$$
$$= \exp. E\frac{d}{d\psi} F(\exp. \psi\sqrt{-1}).$$

Then $$R(f) = \exp. E\frac{d}{d\psi} R(\psi),$$

and for two variables,

$$R(f, f') = \exp. \left(E\frac{d}{d\psi} + E'\frac{d}{d\psi'}\right) R(\psi, \psi'). \quad \ldots(6)$$

Again the eccentricities are concentrated in the factors outside the functional sign.

Owing to the fact that ρ is of the form $1 + \rho_1$, where $|\rho_1| < 1$, the two forms of expansion pointed out in 2·7 are available. The binomial form gives

$$\rho^D = (1 + \rho_1)^D = 1 + \rho_1 D + \frac{\rho_1^2}{2!} D(D-1) + \ldots, \quad \ldots(7)$$

and the exponential form,

$$\rho^D = 1 + \log\rho . D + \frac{(\log\rho)^2}{2!} D^2 + \ldots, \quad \ldots\ldots\ldots(8)$$

where

$$\log\rho = \rho_1 - \tfrac{1}{2}\rho_1^2 + \tfrac{1}{3}\rho_1^3 - \tfrac{1}{4}\rho_1^4 + \ldots,$$
$$(\log\rho)^2 = \rho_1^2 - \rho_1^3 + \tfrac{5}{12}\rho_1^4 - \ldots,$$
$$(\log\rho)^3 = \rho_1^3 - \tfrac{3}{2}\rho_1^4 + \ldots, \quad (\log\rho)^4 = \rho_1^4 + \ldots, \text{ etc.}$$

The latter form is valuable chiefly when the numerical value of e is used, since the coefficients in the functions of ρ are then numerical, and numerical harmonic analysis is efficient for the expansion of the powers of $\log\rho$. Similar remarks apply to the expansion of ρ'^{-D-1} which should be made in the form

$$\rho'^{-D-1} = \frac{1}{\rho'} - \frac{\log\rho'}{\rho'} D + \frac{(\log\rho')^2}{\rho' . 2!} D^2 - \ldots . \quad \ldots\ldots(9)$$

The harmonic analysis is made with the functions

$$1/\rho', \quad \log\rho'/\rho', \ldots$$

4·7. There appears to be no escape from the fact that the development of the disturbing function requires a five-fold series. Developments along powers of e require in reality a double series because r, f require different methods; we may make various combinations of them but the duplicity remains. A similar statement is true of r', f'. The development along powers of the inclination is also double but has been made essentially single by the device of including the factor $\cos^2 \frac{1}{2} I$ in the functions of α (cf. 4·13 (3); also the last paragraph of 4·31). And finally we have the development along powers of α. Out of this six-fold development, one-fold of the development can be avoided by the proper use of the fact that R is a homogeneous function of r, r'. With the methods given in this chapter, the development takes the form of series along powers of the inclination and multiples of the difference of the anomalies, and along powers of the three operators D, B, B', the possibility of such expansions being due to the fact that any given power of these three operators has as a factor of its coefficient the same power of the eccentricities.

4·8. *Expansion of* $1/\Delta$ *along powers of* e, e' *and multiples of* f, f'.

This requires the substitution for r, r' of the expressions,

$$r = \frac{a(1-e^2)}{1+e\cos f}, \quad r' = \frac{a'(1-e'^2)}{1+e'\cos f'}. \quad \text{............}(1)$$

As in 3·3, put $\phi = \exp. f\sqrt{-1}$, $e(1+\eta^2) = 2\eta$, and let

$$r = r_0 \rho, \quad r_0 = a\frac{(1-\eta^2)^2}{1+\eta^2}, \quad \rho = \frac{1}{(1+\eta\phi)(1+\eta/\phi)}, \quad \text{...}(2)$$

with similar expressions for r', r_0', ρ'.

Thence, according to 4·6 (4),

$$\frac{1}{\Delta} = \frac{1}{r_0'} \rho^D \rho'^{-D-1} \frac{1}{\Delta_1}, \quad \text{....................}(3)$$

where $$\Delta_1^2 = 1 + \alpha^2 - 2\alpha \cos S, \quad \alpha = \frac{r_0}{r_0'}, \quad D = \alpha\frac{d}{d\alpha}. \quad \text{...}(4)$$

The expansions of ρ^D, ρ'^{-D-1} into Fourier series with arguments f, f' and coefficients depending on positive integral powers of D

are given by the formulae of 2·16. It will be noticed that ρ^D is equivalent to the function expanded in 2·16 if we put $-\eta, D, D, \phi$ for a, s, t, x, respectively. Hence

$$\rho^D = \Sigma \eta^j \binom{-D}{j} F(D+j,\ D,\ j+1,\ \eta^2) \,.\, 2\cos jf, \quad \ldots(5)$$

with $j=0, 1, 2, \ldots$, the factor 2 being omitted when $j=0$.

Similarly,

$$\rho'^{-D-1} = \Sigma \eta'^{j'} \binom{D+1}{j'} F(-D-1+j', -D-1, j+1, \eta'^2) \,.\, 2\cos j'f'. \quad \ldots\ldots(6)$$

In forming the product of these two series, the rules noted in 2·19 are to be followed. This product, inserted in (3), gives

$$\frac{1}{\Delta} = \frac{1}{r_0'} \Sigma\Sigma \eta^j \eta'^{j'} \binom{-D}{j}\binom{D+1}{j'} FF' \,.\, \frac{1}{\Delta_1} \,.\, 2\cos(jf \pm j'f'), \quad \ldots(7)$$

where F, F' denote the hypergeometric series in (5), (6) respectively.

This is the required expansion. The portions of the coefficient which depend on D have to be expanded in positive powers of D and these are operators acting on $1/\Delta_1$ which contains α only in the explicit form shown in (4). In these expansions it is important to notice that any power of D is always accompanied by at least the same power of η, η', so that the number of powers of D required is the same as the order with respect to the eccentricities to which the expansion is to be developed.

4·9. It is sometimes more convenient to use 2·16 (5) or 2·16 (4) for the developments. The necessary changes are easily seen. If we use the former, the formula 4·8 (7) will still serve if we put

$$r_0 = \frac{a}{1+\eta^2}, \quad r_0' = \frac{a'}{1+\eta'^2}, \quad \alpha = \frac{r_0}{r_0'}, \quad \ldots\ldots\ldots(1)$$

$$F = F(1-D,\ j-D+1,\ j+1,\ \eta^2),$$
$$F' = F(2+D,\ j-D-2,\ j+1,\ \eta^2), \quad \ldots\ldots(2)$$

and multiply the result by $(1-\eta^2)/(1-\eta'^2)$. In adapting the work to this formula, we have put

$$\rho = \frac{(1-\eta^2)^2}{(1+\eta\phi)(1+\eta/\phi)}, \quad \rho' = \frac{(1-\eta'^2)^2}{(1+\eta'\phi')(1+\eta'/\phi')}. \quad \ldots(3)$$

If formula 2·16 (4) is to be used, we put

$$r_0 = a\frac{1-\eta^2}{1+\eta^2} = a\sqrt{1-e^2}, \quad r_0' = a'\frac{1-\eta'^2}{1+\eta'^2} = a'\sqrt{1-e'^2}, \quad \alpha = \frac{r_0}{r_0'}, \qquad \text{......(4)}$$

with

$$F = F\left(1-D,\ D,\ j+1,\ \frac{-\eta^2}{1-\eta^2}\right),$$
$$F' = F\left(2+D,\ -D-1,\ j+1,\ \frac{-\eta'^2}{1-\eta'^2}\right). \quad \text{...(5)}$$

In adapting to this formula, we have used

$$\rho = \frac{1-\eta^2}{(1+\eta\phi)(1+\eta/\phi)}, \quad \rho' = \frac{1-\eta'^2}{(1+\eta'\phi')(1+\eta'/\phi')}. \quad \text{...(6)}$$

4·10. The operator $A = D + \frac{1}{2}$ has one advantage over D which may render its use advisable in some problems. This advantage results from the fact that when the expansion of the operator $\binom{-D}{j} F$ has been made in powers of A, that of $\binom{D+1}{j} F'$ can be immediately written down by changing the sign of A and substituting η' for η.

4·11. The expansion of any function of r, r', Δ can evidently be made by exactly the same methods. We have, for example,

$$\frac{r^p}{r'^{p'}} \cdot \frac{1}{\Delta^s} = \frac{r_0^p}{r_0'^{p+s}} \rho^{D+p} \rho'^{-D-1-p'} \frac{1}{\Delta_1^s}, \quad \text{..............(1)}$$

the expansion of which follows exactly the same plan.

4·12. The complication of the various values of α over that usually used, namely a/a', is more apparent than real, since the numerical value of α is always used. Further, as η is greater than η' in most asteroid problems, and as the convergence is improved by diminishing α, there is an advantage with these values over the value a/a'. The slight disadvantage which arises when we have to differentiate with respect to e or η is easily dealt with by adding to the derivative with respect to η, so far as it occurs explicitly, the derivative $D.\partial\alpha/\alpha\partial\eta$.

4·13. *Development along powers of the inclination.*

In the previous paragraphs $1/\Delta$ has been developed into cosines of multiples of f, f', with coefficients which depend on η, η', $1/\Delta_1$ and on the derivatives of $1/\Delta_1$ with respect to $\log\alpha$. Now

$$\Delta_1^2 = 1 + \alpha^2 - 2\alpha\cos S, \quad \text{..................(1)}$$

where, by 4·1 (3),

$$\cos S = \cos^2 \tfrac{1}{2}I \cos(v - v') + \sin^2 \tfrac{1}{2}I \cos(v + v' - 2\theta). \quad \ldots(2)$$

The general plan requires the expansion of $1/\Delta_1$ into a double Fourier series with arguments $v - v'$, $v + v' - 2\theta$, and this might be achieved by first expanding into a Fourier series with argument S and then expanding $\cos iS$ into sums of cosines of multiples of these two angles. More rapid convergence with less computation can be obtained by making the development depend on the Fourier expansion of $1/\Delta_0$, where

$$\Delta_0^2 = 1 + \alpha^2 - 2\alpha \cos^2 \tfrac{1}{2}I \cos(v - v'), \quad \ldots\ldots\ldots(3)$$

rather than on the same function with $\cos^2 \frac{1}{2}I$ replaced by unity.

With the definition (3) of Δ_0, we have

$$\frac{1}{\Delta_1} = \frac{1}{\Delta_0}\left\{1 - \frac{2\alpha \sin^2 \frac{1}{2}I \cos(v + v' - 2\theta)}{\Delta_0^2}\right\}^{-\frac{1}{2}},$$

which is then expanded by the binomial theorem. This expansion evidently involves odd negative powers of Δ_0 accompanied by even powers of $\sin \frac{1}{2}I$. The powers of $\cos(v + v' - 2\theta)$ are to be expressed as cosines of multiples of the angle. Instead of giving the general form of this expansion, we set it down as far as the eighth power of $\sin \frac{1}{2}I$ which will be sufficient for all practical needs.

Define R_1, R_3, ..., by

$$R_1 = \frac{1}{\Delta_0}, \quad R_{2s} = \frac{\frac{1}{2}\cdot\frac{3}{2}\cdots(s-1)(\alpha \cos^2 \frac{1}{2}I)^{s-\frac{1}{2}}}{\Delta_0^{2s}}, \quad \ldots\ldots(4)$$

where $2s$ takes the values 3, 5, 7, ... We then obtain

$$\begin{aligned}\frac{1}{\Delta_1} &= R_1 + R_5 \tan^4 \tfrac{1}{2}I + \tfrac{1}{4} R_9 \tan^8 \tfrac{1}{2}I \\ &\quad + (2R_3 \tan^2 \tfrac{1}{2}I + R_7 \tan^6 \tfrac{1}{2}I)\cos(v + v' - 2\theta) \\ &\quad + (R_5 \tan^4 \tfrac{1}{2}I + \tfrac{1}{3} R_9 \tan^8 \tfrac{1}{2}I)\cos 2(v + v' - 2\theta) \\ &\quad + \tfrac{1}{3} R_7 \tan^6 \tfrac{1}{2}I \cos 3(v + v' - 2\theta) + \tfrac{1}{12} R_9 \tan^8 \tfrac{1}{2}I \cos 4(v + v' - 2\theta).\end{aligned}$$

$$\ldots\ldots(5)$$

In this development, it should be noticed that a multiple of θ in the angle is always accompanied by at least the same power of $\tan \frac{1}{2}I$ in the coefficient and that the series in any coefficient proceeds by powers of $\tan^4 \frac{1}{2}I$.

The angles v, v' are expressed in terms of the true anomalies by means of the relations

$$v = f + \varpi, \quad v' = f' + \varpi'. \qquad \text{.................. (6)}$$

The development in powers of the eccentricities contained the angles f, f' only, while this development contains the angles

$$f - f' + \varpi - \varpi', \quad f + f' + \varpi + \varpi' - 2\theta.$$

When the functions R_{2s} have been developed as Fourier series, and products of cosines replaced by sums of cosines, we shall have a development containing multiples of the four angles

$$f, \quad f', \quad \varpi - \varpi', \quad \varpi + \varpi' - 2\theta,$$

and this development will have the property that the difference, taken positively, of the multiples of f, f' in any angle will be accompanied by the same power of e, e', I in the coefficient.

4·14. *Development in multiples of f, f' to the third order.*

The development of the R_{2s} is given in a later section of this chapter, (4·23), in the form

$$R_{2s} = \sum_i \beta_s^{(i)} \cos i (f - f' + \varpi - \varpi'), \quad \beta_s^{(-i)} = \beta_s^{(i)}, \quad i = 0, \pm 1, \pm 2, \ldots. \qquad \text{......(1)}$$

The coefficients are functions of α, $\cos^2 \tfrac{1}{2} I$ only, and the operators D^j act solely on these coefficients. The value of α given by 4·8 (2), 4·8 (4) is used here.

By carrying out the various steps outlined above, we obtain the following development as far as the third order with respect to η, η', $\tan \tfrac{1}{2} I$.

$$\frac{r_0'}{\Delta} = (F_1 \Sigma \beta_{\frac{1}{2}}^{(i)} + 2 \tan^2 \tfrac{1}{2} I \,.\, F_3 \Sigma \beta_{\frac{3}{2}}^{(i)}) \cos i (f - f' + \varpi - \varpi'),$$

where

$$\begin{aligned} F_1 = 1 &+ \eta^2 D^2 + \eta'^2 (D+1)^2 \\ &- \{2\eta D + \eta^3 D^2 (D+1) + \eta\eta'^2 D (D+1)^2\} \cos f \\ &+ \{2\eta' (D+1) + \eta'^3 D (D+1)^2 + 2\eta' \eta^2 D^2 (D+1)\} \cos f' \\ &+ (D^2 + D) \{\eta^2 \cos 2f - 2\eta\eta' \cos (f \pm f') + \eta'^2 \cos 2f'\} \\ &- \tfrac{1}{3} \eta^3 (D^3 + 3D^2 + 2D) \cos 3f + \tfrac{1}{3} \eta'^3 (D^3 - D) \cos 3f' \\ &+ \eta^2 \eta' D (D+1)^2 \cos (2f \pm f') - \eta\eta'^2 D^2 (D+1) \cos (f \pm 2f'), \end{aligned}$$

$$F_3 = \cos(f + f' + \Theta) - \eta D \cos(2f + f' + \Theta) - \eta D \cos(f' + \Theta)$$
$$+ \eta'(D+1)\cos(f + 2f' + \Theta) + \eta'(D+1)\cos(f + \Theta),$$
$$\Theta = \varpi + \varpi' - 2\theta.$$

The double sign means that there are two terms each having the coefficient set down.

The final step, that of expressing the products of the cosines in F_1, F_3 by $\cos i(f - f' + \varpi - \varpi')$ as cosines of sums and differences of the angles, is to be carried out. This is equivalent to adding the angle $i(f - f' + \varpi - \varpi')$ to each of the angles in F_1, F_3, because $\beta_s^{(-i)} = \beta_s^{(i)}$. The term in F_1 independent of f, f' requires no treatment; i receives all positive and negative integral values and zero.

4·15. *Transformation from the true to the mean anomalies.*

The development in terms of the true anomalies consists of a sum of terms of the type $A \cos(jf + j'f' + C)$, where A depends on a, a', e, e', I and C on ϖ, ϖ', θ. To transform to a development in which the arguments are functions of the mean anomalies, we make use of the expansions

$$f = g + E = g + \Sigma f_i \sin ig, \quad f' = g' + E' = g' + \Sigma f_i' \sin ig',$$

obtained in 3·11 (6), together with 4·6 (6) which gives

$$\cos(jf + j'f' + C) = \exp.\left(E\frac{\partial}{\partial g} + E'\frac{\partial}{\partial g'}\right)\cos(jg + j'g' + C). \qquad \text{......(1)}$$

The exponential is expanded in powers and products of $E\partial/\partial g$, $E'\partial/\partial g'$, and this requires the expression of powers of E, E' as Fourier series with arguments g, g'; the operators $\partial/\partial g$, $\partial'/\partial g'$ act only on the explicit functions of g, g' and not on E, E'.

The same result is reached by writing

$$\cos(jf + j'f' + C) = \cos(jg + j'g' + C)\cos(jE + j'E')$$
$$- \sin(jg + j'g' + C)\sin(jE + j'E'),$$

and expressing $\quad \begin{matrix}\cos\\ \sin\end{matrix} jE, \quad \begin{matrix}\cos\\ \sin\end{matrix} j'E'$

as Fourier series with arguments g, g' respectively, for the different values of j, j' needed. The calculations of the functions

of E, E' needed can be made in series or numerically by harmonic analysis.

Properties of the expansion. Since R is independent of the directions of the axes of the frame of reference, it is independent of the origin from which the angles used in the expansion are measured. Hence the algebraic sum of the multiples of such angles present in any term is zero.

Thus if w, w' be the mean longitudes, and ϖ, ϖ', θ the longitudes of the perihelia and node, and if any argument in the expansion be

$$iw + i'w' + j\varpi + j'\varpi' + 2h\theta, \qquad \text{..............(2)}$$

we have

$$i + i' + j + j' + 2h = 0.$$

The original form of R was an expression in terms of v, v', r, r', Γ, 2θ. It was pointed out in 3·16, that the expansions of v, r in terms of g or $v - \varpi$ are d'Alembert series as far as the association of powers of e with multiples of ϖ is concerned, and the same is true of v', r' with respect to e', ϖ'. It follows that R has the same properties. Further, the expansion 4·13 (5) shows that R is a d'Alembert series with respect to Γ, 2θ. It follows that the coefficient of a term with the argument (2) is of order $|j| + |j'| + |2h|$ with respect to the eccentricities and inclination.

But
$$|j| + |j'| + |2h| \geqq |j + j' + 2h| = |i + i'|.$$

Hence, the order of the coefficient of any term in the expansion of R is equal to or greater than the algebraic sum of the multiples of w, w' present in that term, and the same is true for the multiples of g, g' when we put $w = g + \varpi$, $w' = g' + \varpi'$.

This property at once gives the lowest order of the coefficient of any term in a numerical expansion of R.

When numerical values of the elements are used, the expression for a'/Δ in terms of the true anomalies may be put into the form

$$\Sigma C_{j,j'} \cos(jf + j'f') + \Sigma S_{j,j'} \sin(jf + j'f'), \qquad j, j' = 0, \pm 1, \ldots,$$

or
$$\Sigma \cos jf\,(C_{j,j'} \cos j'f' + S_{j,j'} \sin j'f')$$
$$+ \Sigma \sin jf\,(C'_{j,j'} \cos j'f' + S'_{j,j'} \sin j'f'), \qquad j, j' = 0, 1, 2, \ldots.$$

The portions in brackets are transformed to multiples of g' by the relation $f' = g' + E'$, either by series or by harmonic analysis.

The series are then re-arranged in the form

$$\Sigma \cos j'g' \, (A_{j,j'} \cos jf + B_{j,j'} \sin jf)$$
$$+ \Sigma \sin j'g' \, (A'_{j,j'} \cos jf + B'_{j,j'} \sin jf), \qquad j, j' = 0, 1, 2, \ldots,$$

and the change to multiples of g is carried out by using the relation $f = g + E$. By following this procedure we can limit the additional work required to obtain the coefficients of the long period terms to a higher degree of accuracy, owing to certain peculiarities in the series for r, f.

4·16. The value of r in terms of f is the series

$$r = a - ae \cos f + \ldots.$$

Actually, this gives series along powers of $\frac{1}{2}e$, because the long period terms always arise from the expression of the product of two cosines as the sum of two cosines; only one of the latter is needed more accurately. The same is true of r'. But when we substitute for f in terms of g by means of the series

$$f = g + 2e \sin g + \ldots,$$

we are substantially expanding in powers of e instead of $\frac{1}{2}e$; the coefficients, which depend on α, are in general of the same order of magnitude for the series giving r in terms of f, and f in terms of g. Further, many of the actual problems are those of asteroids disturbed by one of the great planets and the eccentricities of the orbits of the latter are small. Thus while the steps up to the last have to be carried out to the full degree of accuracy, the series converge rapidly. The convergence is slowest in the last step, but it is here that we can make selection of the terms which have to be accurately computed, the remainder requiring a much lower degree of accuracy.

The calculation of the coefficient of a particular term can also be efficiently carried out by the method which follows.

4·17. *Calculation of the coefficient of a particular term.*

For this calculation we can make use of the theorem of 2·6, where it is shown that the coefficient of $\cos(ig + i'g')$ in the

expansion of $F(f, f')$ is the same as the constant term, when g, g' are expressed in terms of f, f', of one of the expressions

$$2\cos(ig + i'g')\, F \frac{dg}{df}\frac{dg'}{df'}, \quad -\frac{2}{ii'}\cos(ig+i'g')\frac{\partial^2 F}{\partial f \partial f'}, \quad \ldots(1)$$

$$\frac{2}{i}\sin(ig+i'g')\frac{\partial F}{\partial f}\frac{dg'}{df'}, \quad \frac{2}{i'}\sin(ig+i'g')\frac{\partial F}{\partial f'}\frac{dg}{df}. \ldots(2)$$

For the coefficient of $\sin(ig+i'g')$, change cosine to sine in (1) and sine to $-$cosine in (2).

To make use of them we have the relations

$$\frac{dg}{df} = \frac{r^2}{a^2\sqrt{1-e^2}} = \frac{(1-e^2)^{\frac{3}{2}}}{(1+e\cos f)^2}, \quad \ldots\ldots\ldots\ldots(3)$$

$$g = X + e\sin X, \quad \tan\tfrac{1}{2}X = \sqrt{\frac{1-e}{1+e}}\tan\tfrac{1}{2}f, \quad \ldots\ldots(4)$$

with similar expressions for accented letters. An alternative to (3), (4) is the use of the series 3·8 (3).

If we make use of the first of the forms, the initial expansion of $r^2/r'^2\Delta$ instead of the expansion of $1/\Delta$, would be made: the method for doing so is shown in 4·11, and the values $p=2$, $p'=-2$ in 4·11 (1) would be used. With this formula the expansions 4·9 (2) or 4·9 (5) are recommended. In F we replace D by $D+2$ and in F' we replace D by $D-2$: the same changes must of course be made in the binomial coefficients in 4·8 (7).

If, however, the plan is used to get a particular coefficient to a higher degree of accuracy after a general development of $1/\Delta$ has been made, the second form of (1) is of advantage because the development already made will serve; in such cases neither i nor i' is zero, so that this form is always available.

4·18. *Calculation of the constant term.*

This is sometimes needed to a high degree of accuracy. According to the theorem of 2·6 the constant term in the expansion of $1/\Delta$ in multiples of g, g' is the same as the constant term in the expansion of

$$\frac{1}{\Delta}\frac{dg}{df}\frac{dg'}{df'} = \frac{r^2 r'^2}{\Delta \,.\, a^2 a'^2\sqrt{1-e^2}\sqrt{1-e'^2}}$$

in multiples of f, f'. The use of the formula 4·11 (1) with $p=2$, $p'=-2$ is indicated. This expansion does not require the use of the relation connecting f with g or f' with g'; it depends solely on expansions along multiples of f, f', and therefore requires merely the substitution for r, r' of their expressions in terms of f, f'. A literal development to the eighth order with respect to the eccentricities and inclination is to be found* in *Astr. Jour.* vol. 40, pp. 35–38.

4·19. When harmonic analysis is used to obtain functions of f in terms of g and those of f' in terms of g', the computation can be made as follows.

If multiples of g not higher than the sixth are needed the seven special values of g namely, 0°, 30°, 60°, 90°, 120°, 150°, 180°, are recommended. If two more values be needed, those for 45°, 135° can be added, and with two fewer, those for 30°, 150° can be omitted. It is useful to notice that the addition of new values does not require the greater part of the work, which is the computation of the special values of the function, to be done again; only certain small portions of the analysis have to be repeated.

The values of X for the chosen values of g are obtained from Kepler's equation

$$X = g + e \sin X,$$

by one of the methods given in 3·17. From the relation

$$\tan \tfrac{1}{2} f = \sqrt{\frac{1-e}{1+e}} \tan \tfrac{1}{2} X,$$

the special values of f and thence those of any function of f are then obtained. As f, g take the values 0°, 180° together, there are only 3, 5 or 7 special values of f to be computed for each planet. Methods for analysis into Fourier series are given at the end of this volume.

The functions of f needed are $\cos jf$, $\sin jf$ for a number of integral values of j. It is more convenient to calculate $\cos j(f-g)$, $\sin j(f-g)$, and afterwards to deduce the expansions of $\cos jf$, $\sin jf$ by the use of the factors $\cos jg$, $\sin jg$.

Most of the asteroid problems require the calculation of the perturbations by Jupiter and Saturn only. The series for $\cos jf'$, $\sin jf'$, once computed for these two planets will serve for all cases; small changes in the values of the eccentricities are easily made since the power of e which accompanies any term is known by the multiple of g in its argument.

Harmonic analysis is usually so much more accurate with respect to convergence, and is so much more easily controlled than literal expansions, that

* E. W. Brown, *The Expansion of the Constant term of the Disturbing Function to any order.*

it should be used whenever possible. Where many such analyses are to be carried out, a systematic arrangement of the work, by which one operation at a time is performed on all the functions to be analysed, permits the calculations to be carried out rapidly and accurately. See App. A.

4·20. *Development in terms of the eccentric anomalies.*

The expressions in 4·1 give $1/\Delta$ as a function of r, f, r', f', θ, I. Also in 3·3 (8) and (9), with the notation $\phi = \exp. f\sqrt{-1}$, $\chi = \exp. X\sqrt{-1}$, where X is the eccentric anomaly, it has been found that

$$r = \frac{a}{1+\eta^2}(1-\eta/\chi)(1-\eta\chi), \quad \phi = \chi(1-\eta/\chi)/(1-\eta\chi),$$

with similar expressions for r', ϕ'. If we put

$$r_0 = \frac{a}{1+\eta^2}, \quad \rho = (1-\eta/\chi)(1-\eta\chi), \quad \alpha = \frac{r_0}{r_0'}, \quad \phi = \chi p, \quad \phi' = \chi' p',$$

$$D = \alpha\frac{\partial}{\partial\alpha}, \quad B = \chi\frac{\partial}{\partial\chi} = \frac{1}{\sqrt{-1}}\frac{\partial}{\partial X}, \quad B' = \chi'\frac{\partial}{\partial\chi'},$$

we can make use of the theorem of 2·7 with four independent variables.

Now $1/\Delta$ is equal to a function of r/r', ϕ, ϕ' divided by r'. Hence, if $\Delta_X . r_0'$ be what Δ becomes when we replace r, r', ϕ, ϕ' by r_0, r_0', χ, χ', the theorem gives

$$\begin{aligned}\frac{r_0'}{\Delta} &= \rho^D \rho'^{-D-1} p^B p'^{B'} \frac{1}{\Delta_X} \\ &= (1-\eta/\chi)^{D+B}(1-\eta\chi)^{D-B}(1-\eta'/\chi')^{-D-1+B'} \\ &\qquad \times (1-\eta'\chi')^{-D-1-B'}\frac{1}{\Delta_X}. \quad \text{......(1)}\end{aligned}$$

This has to be expanded in powers of the indices.

The expansion of the first pair of factors is made by the theorem of 2·16. It gives

$$\begin{aligned}(1-\eta/\chi)^{D+B}(1-\eta\chi)^{D-B} &= F(-D+B, -D-B, 1, \eta^2) \\ &+ \Sigma\left(-\frac{\eta}{\chi}\right)^j \binom{D+B}{j} F(-D-B+j, -D+B, j+1, \eta^2) \\ &+ \Sigma(-\eta\chi)^j \binom{D-B}{j} F(-D+B+j, -D-B, j+1, \eta^2). \\ &\qquad \text{......(2)}\end{aligned}$$

The result is unchanged if we change the sign of $\sqrt{-1}$, for then χ, $1/\chi$ interchange and also B, $-B$; it is therefore a real Fourier series with argument X.

The product of the second pair of factors is obtained by putting $-D-1$, η', B', χ', j' for D, η, B, χ, j respectively in (2), and it has the same properties.

Since I, θ are present in Δ_X in the same way as they were present in Δ_1, the expansion of Δ_X along powers of the inclination follows the same plan as that of Δ_1 in 4·13. In fact, if we put $v = X + \varpi$, $v' = X' + \varpi'$ in 4·13 (3), 4·13 (5) (taking note of the different significations of r_0, r_0', a), the results can be used here without further change.

Newcomb has given (*Astr. Eph. Papers*, vol. 3) a detailed expansion of the disturbing function in terms of the eccentric anomalies, certain portions of which are taken to the seventh order with respect to the eccentricities. He uses an operator but did not obtain the general formula which permits any coefficient to be written down at once. The latter was given by one of us (E. W. Brown, *Astr. Jour.* vol. 40, p. 19, 1930) in terms of the operator D and certain integers i, i' and later (*Astr. Jour.* vol. 40, p. 61, 1930) in the improved form shown in the text with the use of the operators D, B, B'.

4·21. *Transformation from eccentric to mean anomalies.*

After the disturbing function has been expanded in cosines and sines of multiples of X, X', the transformation to mean anomalies can be effected by the formulae of 3·10, which give

$$\left.\begin{aligned}\cos jX &= A + \Sigma_\kappa \frac{j}{\kappa} J_{\kappa-j}(\kappa e)\cos\kappa g,\\ \sin jX &= \qquad \Sigma_\kappa \frac{j}{\kappa} J_{\kappa-j}(\kappa e)\sin\kappa g,\end{aligned}\right\}\quad \kappa = \pm 1, \pm 2, \ldots \qquad (1)$$

or, if $\zeta = \exp. g\sqrt{-1}$, in the exponential form,

$$\chi^j = A + \Sigma_\kappa \frac{j}{\kappa} J_{(\kappa-j)}(\kappa e)\,.\,\zeta^\kappa, \qquad (2)$$

where $A = 0$ or $-\frac{1}{2}e$ according as $j \neq 1$ or $j = 1$; j may have any positive or negative integral value. A similar set of formulae holds for accented letters.

Since the only terms in the development which give a constant

part are those containing the first multiples of X, X', the constant term of the development in terms of g, g' is obtained by adding to that in terms of X, X', the terms

$$-\tfrac{1}{2}e\,.\,\text{coef. of}\cos X-\tfrac{1}{2}e'\,.\,\text{coef. of}\cos X'$$
$$+\frac{ee'}{4}\,.\,\text{sum of coef. of}\cos(X\pm X').\quad\ldots\ldots(3)$$

The method developed in 4·17 for the calculation of the coefficient of a particular periodic term and that in 4·18 for the constant term can evidently be applied to the transformation from eccentric to mean anomalies. For functions of f, f', we substitute functions of X, X', with

$$g=X-e\sin X,\quad \frac{dg}{dX}=1-e\cos X=\frac{r}{a}.$$

But the coefficients in the expansions of $\cos(jX+j'X')$, $\sin(jX+j'X')$ in terms of g, g' can now be written down in terms of Bessel functions, as defined in 2·14. For both periodic and constant terms, this process is equivalent to that in the text and is merely a different mode of stating it.

4·22. A detailed comparison of the relative advantages of a primary development in terms of the true or eccentric anomalies appears to favour the former. In the first place, the expansion in terms of the true anomalies requires the use of only one operator D, while that in terms of the eccentric anomalies requires the use of three operators D, B, B'. These three operators produce both cosines and sines while, with the operator D alone, only cosines are present, and therefore in reducing products of cosines to sums of cosines, there will always be pairs of coefficients which are the same.

It might be thought that the change from true to mean anomalies is more complicated than that from eccentric to mean, because we cannot use general formulae like the Bessel functions to make the change. As a matter of fact, the actual labour of making the expansions differs very little in the two cases, whether literal or numerical values of the eccentricities be used. The developed series in powers of e have to be used in either case, and such series are available in the tables of Leverrier and Cayley, if a literal expansion is desired. With a numerical expansion by harmonic analysis the only additional work is the calculation, for a few special values, of the values of f, after those of X have been found, from the equation 3·7 (1).

A point connected with the rate of convergence along powers of e, and rarely mentioned, deserves some stress because the work of calculating the coefficient of some particular term to an order of accuracy higher than that of the general development can be made lighter by taking it into consideration. It has been pointed out in 4·16 that the rates of convergence of the

series for r, r' in terms of f, f' are more rapid than those of f, f' in terms of g, g'. There is no such difference in the rates of convergence in passing directly from r, f, r', f' to X, X' and from X, X' to g, g'. Thus the longer development in terms of X, X' must also have the full accuracy desired while the shorter development in terms of f, f' is still more abbreviated by the separation of the more rapidly converging series from the more slowly converging series.

The advantage possessed by the expansion in terms of the eccentric anomalies in the form given in the text, consists in the fact that it is the only method known by which any coefficient in the development of the disturbing function can be written down from a general formula; it contains the operators D, B, B', and the Bessel functions. The highest power of these operators present in any portion of a coefficient is the same as the order of that coefficient with respect to the eccentricities, so that stoppage at a given power of the latter involves stoppage at the same power of the former. The order of a Bessel function is known from its suffix. But the formula suffers from the defect pointed out in the paragraph following 3·11 (6) for the case of the general expansion of f in terms of g, namely, that $j+1$ numerical coefficients have to be added together to obtain any part of order j; this defect becomes serious when j is large.

4·23. *The functions of the major axes.*

The development of the previous sections of this chapter require the calculation of the coefficients $\beta_s^{(i)}$ defined* by

$$R_{2s} = \frac{\frac{1}{2}\cdot\frac{3}{2}\dots(s-1)\,(\alpha\cos^2\frac{1}{2}I)^{s-\frac{1}{2}}}{(1+\alpha^2-2\alpha\cos^2\frac{1}{2}I\cos\psi)^s} = \sum_i \beta_s^{(i)}\cos i\psi, \quad \dots(1)$$

where $i = 0, \pm 1, \pm 2, \dots$; $2s = 1, 3, 5, \dots$; $\beta_s^{(-i)} = \beta_s^{(i)}$;

and for $2s = 1$, the numerator of the fraction is unity. The definitions of ψ, α are immaterial to the work of this section provided $|\alpha| < 1$. We shall need also the derivatives of the coefficients with respect to α or to $\log\alpha$.

It has been pointed out in 4·2 that the magnitude of α in general prevents the use of literal series in powers of α as a practical method for calculation and we must consequently use other devices. In the following paragraphs, transformations of the series in powers of α are made for two of the coefficients so

* This definition of the coefficients without the factor $\frac{1}{2}$ is so much more convenient than that of Leverrier and others who have followed him that we have retained it throughout. See 2·19.

that they may be easily and rapidly obtained. It is then shown how all the remaining coefficients and their derivatives can be deduced from these two by the use of finite formulae. The two coefficients to be first found will be those for $s=\frac{1}{2}$, $i=10, 11$, for reasons which will appear. The more usual plan has been to calculate

$$\beta_{\frac{1}{2}}^{(0)}=\frac{2}{\pi}F_1,\quad \beta_{\frac{1}{2}}^{(1)}=\frac{2}{\pi}(F_1-E_1), \quad \ldots\ldots\ldots\ldots(2)$$

where F_1, E_1 are the elliptic integrals of the first and the second kind, from the tables of Legendre, with $\cos^2\frac{1}{2}I=1$. The tables of Runkle (*Smithsonian Contributions*, 1855) give certain of these coefficients for different values of α; those of Brown and Brouwer (Camb. Univ. Press, 1932) have higher accuracy.

4·24. *The series for $\beta_s^{(i)}$.*

Define α_1, κ by*

$$\frac{\alpha_1}{1+\alpha_1^2}=\frac{\alpha\cos^2\frac{1}{2}I}{1+\alpha^2},\quad \kappa=\cos^2\tfrac{1}{2}I, \quad \ldots\ldots\ldots\ldots(1)$$

so that

$$(1+\alpha^2-2\alpha\cos^2\tfrac{1}{2}I\cos\psi)^{-s}=\left(\frac{\alpha_1}{\alpha\kappa}\right)^s(1+\alpha_1^2-2\alpha_1\cos\psi)^{-s}. \ldots(2)$$

The last factor may be expanded into a Fourier series by the method given in 2·16. By inserting this expansion in 4·23 (1) we obtain

$$R_{2s}=\tfrac{1}{2}\cdot\tfrac{3}{2}\ldots(s-1)(\alpha\kappa)^{s-\frac{1}{2}}\left(\frac{\alpha_1}{\alpha\kappa}\right)^s\Sigma\binom{-s}{i}(-\alpha_1)^iF\,.\,2\cos i\psi$$

$$=\alpha_1^s(\alpha\kappa)^{-\frac{1}{2}}\Sigma\frac{\frac{1}{2}\cdot\frac{3}{2}\ldots(s+i-1)}{i!}\alpha_1^iF\,.\,2\cos i\psi, \quad \ldots\ldots(3)$$

where F is the hypergeometric series given by

$$F=F(s+i,s,i+1,\alpha_1^2)$$

$$=1+\frac{s}{1}\cdot\frac{s+i}{i+1}\alpha_1^2+\frac{s(s+1)}{1\,.\,2}\cdot\frac{(s+i)(s+i+1)}{(i+1)(i+2)}\alpha_1^4+\ldots, \ldots(4)$$

and i takes the values 0, 1, 2, ..., the constant term under the sign of summation in (3)—that for $i=0$—being $F(s,s,1,\alpha_1^2)$.

* The value of α_1 can be readily found by putting $\alpha_1=\tan A_1$, $\alpha=\tan A$, and finding A_1 from $\sin 2A_1=\kappa\sin 2A$.

By means of the transformation 2·15 (3) we have

$$F(s+i, s, i+1, \alpha_1^2) = (1-\alpha_1^2)^{-s} F(1-s, s, i+1, -p),$$

where $$p = \alpha_1^2/(1-\alpha_1^2), \quad \text{.....................(5)}$$

so that

$$\beta_s^{(i)} = \frac{\frac{1}{2}\cdot\frac{3}{2}\ldots(s+i-1)}{i\,!}\,\frac{\alpha_1^{i+s}}{(1-\alpha_1^2)^s}\,\frac{1}{(\alpha\kappa)^{\frac{1}{2}}}\,F(1-s, s, i+1, -p), \quad \text{......(6)}$$

in which

$$F(1-s, s, i+1, -p)$$
$$= 1 + \frac{s}{1}\cdot\frac{s-1}{i+1}\,p + \frac{s(s+1)}{1\,.\,2}\cdot\frac{(s-1)(s-2)}{(i+1)(i+2)}\,p^2 + \ldots. \quad \text{...(7)}$$

This last form of the hypergeometric series is evidently useful for large values of i, since in this case the earlier coefficients of powers of p diminish rapidly. It is true that the series converges only when $|p| < 1$, that is, when $\alpha_1 < 2^{-\frac{1}{2}} = {\cdot}707$, and that the values of the coefficients are sometimes needed for values of α_1 larger than this. But we know that the function which the series represents has no singularity provided $|\alpha_1| < 1$, that is, provided p be finite. It is therefore permissible to use the method of analytic continuation to obtain expansions in powers of $p - p_0$ where $p_0 \neq 0$.

4·25. *The calculation of* a_{10}, a_{11}.

For brevity, let us put $\beta_{\frac{1}{2}}^{(i)} = a_i$, so that

$$(1+\alpha^2 - 2\alpha\cos^2\tfrac{1}{2}I\cos\psi)^{-\frac{1}{2}} = a_0 + 2\Sigma a_i \cos i\psi, \quad \text{...(1)}$$

$$a_i = \left\{\frac{\alpha_1}{\alpha\kappa(1-\alpha_1^2)}\right\}^{-\frac{1}{2}} \alpha^i\,\frac{1\,.\,3\ldots(2i-1)}{2\,.\,4\ldots 2i}\,F(\tfrac{1}{2}, \tfrac{1}{2}, i+1, -p), \quad \text{......(2)}$$

with

$$F = 1 - \frac{1^2}{2\,.(2i+2)}\,p + \frac{1^2\,.\,3^2}{2\,.\,4\,.(2i+2)(2i+4)}\,p^2 - \ldots \quad \text{......(3)}$$

The transformation of (3) to series expanded in powers of $p - p_0$ can be effected by the use of Taylor's series:

$$F(p) = F(p_0) + (p-p_0)\,F'(p_0) + \frac{(p-p_0)^2}{2\,!}\,F''(p_0) + \ldots.$$

The values of $F(p_0)$, $F'(p_0)$ are obtained directly from the series, and the remaining derivatives from the recurrence formula

2·15(5) deduced from the differential equation satisfied by the hypergeometric series*.

On putting $(1-\alpha_1^2)^{\frac{1}{2}} = \alpha_1 p^{-\frac{1}{2}}$, we have the following expressions for the cases $p_0 = 0, \frac{1}{2}, 1$. The factors 46,189 and 176,358 are the respective products 11 . 13 . 17 . 19 and 2 . 13 . 17 . 19 . 21.

$a_{10} = 46{,}189 \sqrt{\dfrac{\alpha_1 p}{\kappa a}} \left(\dfrac{\alpha_1}{4}\right)^9$ multiplied by one of the series,

+1·00000 000	+0·98913 047	+0·97912 120
− ·02272 727 p	− ·02082 065 $(p-\frac{1}{2})$	− ·01926 520 $(p-1)$
+ ·00213 068 p^2	+ ·00171 01 $(p-\frac{1}{2})^2$	+ ·00141 681 $(p-1)^2$
− ·00034 15 p^3	− ·00023 04 $(p-\frac{1}{2})^3$	− ·00016 594 $(p-1)^3$
+ ·00007 47 p^4	+ ·00004 1 $(p-\frac{1}{2})^4$	+ ·00002 515 $(p-1)^4$
− ·00002 0 p^5	− ·00000 9 $(p-\frac{1}{2})^5$	− ·00000 451 $(p-1)^5$
+ ·00000 6 p^6	+ ·00000 2 $(p-\frac{1}{2})^6$	+ ·00000 091 $(p-1)^6$
		− ·00000 020 $(p-1)^7$
		+ ·00000 005 $(p-1)^8$
		− ·00000 001 $(p-1)^9$

$a_{11} = 176{,}358 \sqrt{\dfrac{\alpha_1 p}{\kappa a}} \left(\dfrac{\alpha_1}{4}\right)^{10}$ multiplied by one of the series,

+1·00000 000	+0·99000 356	+0·98074 527
− ·02083 333 p	− ·01920 796 $(p-\frac{1}{2})$	− ·01786 420 $(p-1)$
+ ·00180 288 p^2	+ ·00146 840 $(p-\frac{1}{2})^2$	+ ·00123 022 $(p-1)^2$
− ·00026 83 p^3	− ·00018 54 $(p-\frac{1}{2})^3$	− ·00013 597 $(p-1)^3$
+ ·00005 48 p^4	+ ·00003 11 $(p-\frac{1}{2})^4$	+ ·00001 956 $(p-1)^4$
− ·00001 4 p^5	− ·00000 7 $(p-\frac{1}{2})^5$	− ·00000 335 $(p-1)^5$
+ ·00000 1 p^6	+ ·00000 2 $(p-\frac{1}{2})^6$	+ ·00000 065 $(p-1)^6$
		− ·00000 014 $(p-1)^7$
		+ ·00000 003 $(p-1)^8$
		− ·00000 001 $(p-1)^9$

The series give a_{10}, a_{11} to eight significant figures when $\alpha_1 < \cdot 82$. The table

$\alpha_1 = 0$	·45	·61	·66	·71	·78	·82	·88
$p = 0$	·25	·50	·75	1·00	1·50	2·00	3·00

indicates the series to choose for any given value of α_1. For $\alpha_1 = \cdot 88$, the error is about one part in 10^5.

* E. W. Brown, *Mon. Not. R.A.S.* vol. 88, pp. 459–465. The numerical series given in the text for $p_0 = \frac{1}{2}$, 1 are taken from this paper. Extensions to the cases $p_0 = 2, 3, 4$ are to be found in *Mon. Not.* vol. 92, pp. 224–7.

Alternative form for a_{10}, a_{11}. The ratios of consecutive coefficients of these series all tend to the limit unity, but for large values of i the approach is very slow, and after the first two or three terms the ratio changes slowly. We can make use of this fact by expressing the series $a_0+a_1p+a_2p^2+\ldots$ in the form

$$\{a_0+(a_1-\lambda a_0)\,p+(a_2-\lambda a_1)\,p^2+\ldots\}\div(1-\lambda p),$$

with a suitable choice of λ. In this way the following expressions have been obtained:

$$a_{10}=\cdot 35239{,}4104\left(\frac{p\alpha_1}{\kappa\alpha}\right)^{\frac{1}{2}}\alpha_1^{9}\left(1-\tfrac{1}{44}p+\delta a_{10}\right),$$

$$a_{11}=\cdot 38183{,}0736\left(\frac{p\alpha_1}{\kappa\alpha}\right)^{\frac{1}{2}}\alpha_1^{10}\left(1-\tfrac{1}{48}p+\delta a_{11}\right),$$

where
$$\delta a_{10}=(2131+245p-19p^2)\,p^2\,.\,10^{-6}\div(1+\cdot 275p),$$
$$\delta a_{11}=(1803+183p-12p^2)\,p^2\,.\,10^{-6}\div(1+\cdot 250p).$$

These give a_{10}, a_{11} to six significant figures when $p<1$. Eight significant figures for $p<1$ are furnished by the expressions,

$$\delta a_{10}=\frac{3}{11\cdot 2^7}p^2-\cdot 00034{,}146p^3+\cdot 00007{,}469p^4 \\ -\left(2017+\frac{373p-p^2}{1+\cdot 270p}\right)\frac{p^5\,10^{-8}}{1+\frac{1}{2}p},$$

$$\delta a_{11}=\frac{3}{13\cdot 2^7}p^2-\cdot 00026{,}829p^3+\cdot 00005{,}477p^4 \\ -\left(1387+\frac{204p}{1+\frac{2}{9}p}\right)\frac{p^5\,10^{-8}}{1+\frac{4}{9}p}.$$

4·26. *Formulae for calculating the $\beta_s^{(i)}$ when two consecutive coefficients are known.*

The procedure which appears to give numerical results most easily requires the use of the following formulae. It is to be noticed that as soon as two consecutive coefficients have been found, there is no further need of α_1; the formulae involve only α, κ in the form

$$\epsilon=\frac{1}{\kappa}\left(\frac{1}{\alpha}+\alpha\right). \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

The proofs are given in the sections which follow.

For finding the remaining $a_i=\beta_{\frac{1}{2}}^{(i)}$ when two of them are known, we have the formula,

$$a_i=\epsilon a_{i+1}-a_{i+2}+\frac{1}{2i+1}(\epsilon a_{i+1}-2a_{i+2})\ldots\ldots\ldots(2)$$

for values of i lower than those known, and

$$a_{i+2} = \epsilon a_{i+1} - a_i - \frac{1}{2i+3}(\epsilon a_{i+1} - 2a_i) \quad \ldots\ldots\ldots(3)$$

for higher values of i. These formulae are used by putting $i = 9, 8, \ldots$ successively in (2) when a_{10}, a_{11} have been found; and $i = 10, 11, \ldots$ as far as they are needed in (3). They are deduced from the general formula,

$$\beta_s^{(i)} = \frac{i+1}{i+s}\,\epsilon\beta_s^{(i+1)} - \frac{i-s+2}{i+s}\,\beta_s^{(i+2)}, \quad \ldots\ldots\ldots(4)$$

by giving to s the value $\frac{1}{2}$.

The values of two consecutive coefficients for other values of s are found by putting $s = \frac{1}{2}, \frac{3}{2}, \ldots$, successively in the formulae

$$\left.\begin{aligned} \tfrac{1}{2}\beta_{s+1}^{(i)} + \tfrac{1}{2}\beta_{s+1}^{(i+1)} &= \frac{(2i+2s)\,\beta_s^{(i)} - (2i+2-2s)\,\beta_s^{(i+1)}}{4(\epsilon-2)}, \\ \tfrac{1}{2}\beta_{s+1}^{(i)} - \tfrac{1}{2}\beta_{s+1}^{(i+1)} &= \frac{(2i+2s)\,\beta_s^{(i)} + (2i+2-2s)\,\beta_s^{(i+1)}}{4(\epsilon+2)}. \end{aligned}\right\} \ldots(5)$$

The remaining coefficients are found rapidly from the formula

$$\beta_{s+1}^{(i-1)} = i\beta_s^{(i)} + \beta_{s+1}^{(i+1)}, \quad \ldots\ldots\ldots\ldots\ldots\ldots(6)$$

which may be used backwards or forwards along values of i.

Sufficient checks on the numerical work are obtained in the following manner. When $a_9, a_8, \ldots, a_0$ have been successively computed by means of (2), the value of a_0 can be obtained directly from 4·23 (2) with the use of Legendre's tables: this, in effect, tests the whole series of a_i.

When $\beta_{\frac{3}{2}}^{(11)}, \ldots, \beta_{\frac{3}{2}}^{(0)}$ have been found from (5) for the first two and from (6) with $s = \frac{1}{2}$ for the remaining coefficients, the values of $\beta_{\frac{3}{2}}^{(1)}, \beta_{\frac{3}{2}}^{(0)}$ can be tested by computing from (5) with $s = \frac{1}{2}$, $i = 0$. A similar procedure tests the values for $s = \frac{5}{2}, \frac{7}{2}, \ldots$.

In general, there is a loss of less than one significant figure in running down from $\beta_s^{(10)}$ to $\beta_s^{(0)}$. There is some loss of accuracy in the use of the formula (5), but this loss is balanced by the fact that the higher values of s are present only with the higher powers of the inclination and indirectly of the eccentricities.

4·27. *Formulae for the derivatives of the* $\beta_s^{(i)}$.

The first derivative is obtained from

$$D\beta_s^{(i)} = \frac{1}{\kappa}\left(\frac{1}{\alpha} - \alpha\right)\beta_{s+1}^{(i)} - \tfrac{1}{2}\beta_s^{(i)}, \quad D = \alpha\frac{d}{d\alpha} \quad \ldots\ldots(1)$$

for all values of i with $s = \frac{1}{2}$, and for two consecutive values of i with $s = \frac{3}{2}, \frac{5}{2}, \ldots$, the remaining derivatives being found most easily from

$$D\beta_{s+1}^{(i-1)} = i\,D\beta_s^{(i)} + D\beta_{s+1}^{(i+1)}. \quad \ldots\ldots\ldots\ldots\ldots(2)$$

For the higher derivatives, either of the following formulae in which the index (i), being the same throughout, is omitted, may be used:

$$D^{j+2}\beta_s = \frac{1}{\kappa\alpha}(D-1)^{j+1}\beta_{s+1} - \frac{\alpha}{\kappa}(D+1)^{j+1}\beta_{s+1} - \tfrac{1}{2}D^{j+1}\beta_s, \quad \ldots\ldots(3)$$

$$D^{j+2}\beta_s = (2s-1)\,D^{j+1}\beta_s + \frac{4s\alpha}{\kappa}(D+1)^j\beta_{s+1} + \{i^2 - (s-\tfrac{1}{2})^2\}\,D^j\beta_s + 4\left(1 - \frac{1}{\kappa^2}\right)D^j\beta_{s+2}. \quad \ldots(4)$$

To j are given successively the values 0, 1, 2, ...: the latter formula requires the calculation of one fewer set of coefficients β_s for a given degree of accuracy with respect to the eccentricities (cf. 4·31). These formulae are used like those which precede it. The derivatives for $s = \frac{1}{2}$ and all needed values of i are computed from either (3) or (4). For the remaining values of s, they are used for the computation of the derivatives for the two highest values of i only, the remaining derivatives being found more easily from

$$D^j\beta_{s+1}^{(i-1)} = i\,D^j\beta_s^{(i)} + D^j\beta_{s+1}^{(i+1)}. \quad \ldots\ldots\ldots\ldots\ldots(5)$$

These alternative methods of computation furnish obvious checks.

4·28. The proofs of the preceding formulae are obtained by treating the fundamental expansion,

$$c_s(1 + \alpha^2 - 2\kappa\alpha\cos\psi)^{-s} = (\alpha\kappa)^{\frac{1}{2}-s}\,\Sigma\beta_s^{(i)}\cos i\psi, \quad \ldots(1)$$

where $\qquad c_s = \frac{1}{2}\cdot\frac{3}{2}\ldots(s-1), \qquad i = 0, \pm 1, \pm 2, \ldots,$

as an identity.

The derivative of (1) with respect to ψ gives

$$-sc_s(1+\alpha^2-2\kappa\alpha\cos\psi)^{-s-1}\,.\,2\kappa\alpha\sin\psi = -(\alpha\kappa)^{\frac{1}{2}-s}\,\Sigma\beta_s^{(i)}\,i\sin i\psi. \qquad \text{......(2)}$$

Replace s by $s+1$ in (1) and insert the result in the left-hand member of (2). Since $c_{s+1}=sc_s$, we obtain

$$2\sin\psi\,\Sigma\beta_{s+1}^{(i)}\cos i\psi = \Sigma\beta_s^{(i)}\,i\sin i\psi.$$

The left-hand member of this equation may be expressed as a sum of sines of multiples of ψ. Equating the coefficients of $\sin i\psi$, we obtain

$$\beta_{s+1}^{(i-1)} - \beta_{s+1}^{(i+1)} = i\beta_s^{(i)}, \qquad \text{..................(3)}$$

which is the formula 4·26 (6). The derivatives of this give 4·27 (2), 4·27 (5).

Again, multiply (2) by $1+\alpha^2-2\kappa\alpha\cos\psi$ and insert (1) in the left-hand member of the result. After some reduction, we obtain

$$\kappa\alpha\,\Sigma(i+s)\,\beta_s^{(i)}\sin(i+1)\psi + \kappa\alpha\,\Sigma(i-s)\,\beta_s^{(i)}\sin(i-1)\psi = (1+\alpha^2)\,\Sigma i\beta_s^{(i)}\sin i\psi.$$

In each of these series i takes all integral values from $+\infty$ to $-\infty$. The selection of the coefficients of $\sin(i+1)\psi$, from each of them, gives

$$\kappa\alpha\,(i+s)\,\beta_s^{(i)} + \kappa\alpha\,(i+2-s)\,\beta_s^{(i+2)} = (1+\alpha^2)(i+1)\,\beta_s^{(i+1)},$$

which is the same as 4·26 (4) when we put $\epsilon=(1+\alpha^2)/\kappa\alpha$.

Finally, the identity

$$(1+\alpha^2-2\kappa\alpha\cos\psi)\,\Sigma\,\beta_{i+1}^{(i)}\cos i\psi = s\alpha\kappa\,\Sigma\,\beta_s^{(i)}\cos i\psi,$$

obtained from (1), with the same equation when $s+1$ is put for s, yields by the same procedure as before,

$$(1+\alpha^2)\,\beta_{s+1}^{(i)} - \alpha\kappa\,(\beta_{s+1}^{(i-1)}+\beta_{s+1}^{(i+1)}) = s\alpha\kappa\,\beta_s^{(i)}. \qquad \text{......(4)}$$

If we successively eliminate $\beta_{s+1}^{(i-1)}, \beta_{s+1}^{(i+1)}$ between this equation and (3), we obtain

$$(1+\alpha^2)\,\beta_{s+1}^{(i)} - 2\alpha\kappa\,\beta_{s+1}^{(i+1)} = \alpha\kappa\,(s+i)\,\beta_s^{(i)},$$

$$(1+\alpha^2)\,\beta_{s+1}^{(i)} - 2\alpha\kappa\,\beta_{s+1}^{(i-1)} = \alpha\kappa\,(s-i)\,\beta_s^{(i)}.$$

Change i into $i-1$ in the former of these equations and add the result to and subtract it from the latter; the two equations thus obtained are the same as 4·26 (5).

4·29. The values of the derivatives are obtained as follows. The derivative with respect to α of the logarithm of 4·23 (1) gives, after multiplication by α,

$$\frac{DR_{2s}}{R_{2s}} = s - \tfrac{1}{2} - \frac{2s(\alpha^2 - \kappa\alpha\cos\psi)}{1+\alpha^2 - 2\kappa\alpha\cos\psi}$$

$$= -\tfrac{1}{2} + \frac{s(1-\alpha^2)}{1+\alpha^2 - 2\kappa\alpha\cos\psi}.$$

But, from the definition of R_{2s},

$$R_{2s+2} = \frac{sR_{2s}}{1+\alpha^2 - 2\kappa\alpha\cos\psi}\cdot\kappa\alpha.$$

Hence
$$DR_{2s} = -\tfrac{1}{2}R_{2s} + \frac{1-\alpha^2}{\kappa\alpha}R_{2s+2}.$$

From this equation, by replacing R_{2s}, R_{2s+2} by their expansions in Fourier series and equating the coefficients of $\cos i\psi$, we obtain

$$D\beta_s^{(i)} = \frac{1}{\kappa}\left(\frac{1}{\alpha} - \alpha\right)\beta_{s+1}^{(i)} - \tfrac{1}{2}\beta_s^{(i)},$$

which is the equation 4·27 (1).

The application of the operator D^j to this with the help of the general theorem,

$$D^j\{\alpha^q f(\alpha)\} = \alpha^q (D+q)^j . f(\alpha), \quad \text{..........(1)}$$

furnishes the equation 4·27 (3).

4·30. The proof of the relation 4·27 (4) is more difficult. If we put

$$\rho = 1 + \alpha^2 - 2\alpha\kappa\cos\psi, \quad \text{..................(1)}$$

we have

$$(D\rho)^2 + \left(\frac{\partial\rho}{\partial\psi}\right)^2 = (2\alpha^2 - 2\alpha\kappa\cos\psi)^2 + 4\alpha^2\kappa^2\sin^2\psi$$

$$= 4\alpha^2\rho + 4\alpha^2(\kappa^2 - 1),$$

$$D^2\rho + \frac{\partial^2\rho}{\partial\psi^2} = 4\alpha^2 - 2\alpha\kappa\cos\psi + 2\alpha\kappa\cos\psi = 4\alpha^2.$$

Also
$$D^2\rho^{-s} = s(s+1)(D\rho)^2 . \rho^{-s-2} - sD^2\rho . \rho^{-s-1},$$

$$\frac{\partial^2}{\partial\psi^2}\rho^{-s} = s(s+1)\left(\frac{\partial\rho}{\partial\psi}\right)^2 . \rho^{-s-2} - s\frac{\partial^2\rho}{\partial\psi^2} . \rho^{-s-1}.$$

Whence, with the aid of the two previous equations,

$$\left(D^2+\frac{\partial^2}{\partial\psi^2}\right)\rho^{-s}=s(s+1)\{4\alpha^2\rho+4\alpha^2\kappa^2\}\rho^{-s-2}-4s\alpha^2\rho^{-s-1}$$
$$=4\alpha^2s^2\rho^{-s-1}+4\alpha^2s(s+1)(\kappa^2-1)\rho^{-s-2}. \quad \ldots\ldots(2)$$

Now, the definition (1) and 4·23 (1) give

$$\rho^{-s}=(\alpha\kappa)^{\frac{1}{2}-s}R_{2s}\div c_s, \quad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

and the theorem 4·29 (1), applied to this, gives

$$D^2\rho^{-s}=(\alpha\kappa)^{\frac{1}{2}-s}(D+\tfrac{1}{2}-s)^2R_{2s}\div c_s$$

Insert this result in the left-hand member of (2) and eliminate ρ from the remaining terms of (2) by the use of (3) after replacing s by $s+1$, $s+2$, therein. With the help of the relations

$$c_{s+1}=sc_s, \quad c_{s+2}=s(s+1)c_s,$$

we obtain, after division by suitable factors,

$$\left\{(D+\tfrac{1}{2}-s)^2+\frac{\partial^2}{\partial\psi^2}\right\}R_{2s}=\frac{4\alpha}{\kappa}sR_{2s+2}+4\left(1-\frac{1}{\kappa^2}\right)R_{2s+4}.$$

The final step is the insertion of the Fourier series and the equating of the coefficients of $\cos i\psi$. This process gives

$$\{(D+\tfrac{1}{2}-s)^2-i^2\}\beta_s^{(i)}=\frac{4\alpha}{\kappa}s\beta_{s+1}^{(i)}+4\left(1-\frac{1}{\kappa^2}\right)\beta_{s+2}^{(i)},$$

which is easily seen to be the same as equation 4·27 (4) when we put $j=0$ in the latter. The result may be written

$$D^2\beta_s^{(i)}=\frac{4\alpha}{\kappa}s\beta_{s+1}^{(i)}+(2s-1)D\beta_s^{(i)}+4\left(1-\frac{1}{\kappa^2}\right)\beta_{s+2}^{(i)}$$
$$+\{i^2-(s-\tfrac{1}{2})^2\}\beta_s^{(i)}.$$

The application of the operator D^j to this equation and the use of the theorem 4·29 (1) give 4·27 (4).

4·31. The statement in 4·27 contrasting the formulae 4·27 (3), 4·27 (4), may be justified as follows. The continued use of 4·27 (4) makes $D^j\beta_s$ depend on the calculation of $\beta_{s+j}, \beta_{s+j-1}, \ldots, \beta_s$. If we put $j=0$ in this equation, the right-hand member depends on $D\beta_s$, β_{s+1} which require the calculation of β_{s+1}, β_s only; the

factor of β_{s+2} is $-4\tan^2\frac{1}{2}I(1+\sec^2\frac{1}{2}I)$. Since the operator D^2 is always accompanied by the square of the eccentricities in the development of the disturbing function, the effect of this last term is of the fourth order. The argument for values of j greater than zero in the formula is similar.

Incidentally, this formula shows why a considerable increase in the convergence along powers of the inclination is obtained by the insertion of the factor κ. In general, the coefficients β_{s+j} tend to increase with j for a given value of i and the factor 4 which occurs in this term shows that it will modify the values of the derivatives considerably when I is large. The additional computation caused by its presence is very small*.

4·32. *The literal expansion of* $1/\Delta$ *to the second order, in terms of the mean anomalies.*

This expansion is obtained from that in terms of the true anomalies given in detail in 4·14. The latter contains products of cosines which are expressed as sums of cosines as explained in that paragraph. In writing the result out to the second order, the notation

$$\beta_{\frac{1}{2}}^{(i)}=a_i=a_{-i},\quad \beta_{\frac{3}{2}}^{(i)}=b_i=b_{-i},\quad f-f'+\varpi-\varpi'=v-v',$$

will be used. The result is

$\dfrac{r_0'}{\Delta}=$ sum for all positive and negative values of i, including zero, of

$$\begin{aligned}\{1+\eta^2D^2+\eta'^2(D+1)^2\}\,&a_i\cos i(v-v')\\ &-2\eta\eta'(D^2+D)\,a_i\cos(iv-iv'+f-f')\\ &-2\eta\,Da_i\cos(iv-iv'+f)+2\eta'(D+1)\,a_i\cos(iv-iv'+f')\\ &+\eta^2(D^2+D)\,a_i\cos(iv-iv'+2f)\\ &+\eta'^2(D^2+D)\,a_i\cos(iv-iv'+2f')\\ &-2\eta\eta'(D^2+D)\,a_i\cos(iv-iv'+f+f')\\ &+2\tan^2\tfrac{1}{2}I\,.\,b_i\cos(iv-iv'+v+v'-2\theta).\quad\text{..............(1)}\end{aligned}$$

* These modifications of the usual formulae in which $\kappa=1$, were given by E. W. Brown, *Mon. Not. R.A.S.* vol. 88, pp. 459–465.

The transformation to mean anomalies is made by means of the theorem 4·15 (1), namely,

$$\cos(jf+j'f'+C)=\exp.\left(E\frac{\partial}{\partial g}+E'\frac{\partial}{\partial g'}\right)\cos(jg+j'g'+C),$$

where $$f=g+E,\quad f'=g'+E'.$$

The expression for E is given in 3·16. Expressed in terms of η as far as the second order, it is

$$E=4\eta\sin g+5\eta^2\sin 2g,$$

and we have a similar form for E'. From these we deduce

$$E^2=8\eta^2-8\eta^2\cos 2g,\quad EE'=8\eta\eta'\{\cos(g-g')-\cos(g+g')\}.$$

With the aid of these formulae and of the expansion

$$\exp.\left(E\frac{\partial}{\partial g}+E'\frac{\partial}{\partial g'}\right)=1+E\frac{\partial}{\partial g}+E'\frac{\partial}{\partial g'}$$
$$+\tfrac{1}{2}E^2\frac{\partial^2}{\partial g^2}+\tfrac{1}{2}E'^2\frac{\partial^2}{\partial g'^2}+EE'\frac{\partial^2}{\partial g\partial g'},$$

we obtain the following development:

$$\frac{1}{\Delta}=\frac{1}{r_0'}\times\text{sum for } i=0,\ \pm 1,\ \pm 2,\ \ldots\text{ of}$$

$$\begin{aligned}\{1&+\eta^2(D^2+4D-4i^2)+\eta'^2(D^2-2D-4i^2-3)\}\,a_i\cos(iw-iw')\\ &-2\eta\eta'(D^2+D-4i^2-2i)\,a_i\cos(iw-iw'+g-g')\\ &+2\eta(-D+2i)\,a_i\cos(iw-iw'+g)\\ &+2\eta'(D+1-2i)\,a_i\cos(iw-iw'+g')\\ &+\eta^2(D^2-4iD-3D+4i^2+5i)\,a_i\cos(iw-iw'+2g)\\ &+\eta'^2(D^2-4iD+5D+4i^2-9i+4)\,a_i\cos(iw-iw'+2g')\\ &-2\eta\eta'(D^2-4iD+D+4i^2-2i)\,a_i\cos(iw-iw'+g+g')\\ &+2\tan^2\tfrac{1}{2}I\,.\,b_i\cos(iw-iw'+w+w'-2\theta),\ldots\ldots\ldots\ldots\ldots(2)\end{aligned}$$

in which the notations

$$w,\ w'=\text{mean longitudes}=g+\varpi,\ g'+\varpi'$$

are used.

For convenient reference, we repeat the significations of the remaining symbols used in this development:

$$r_0' = \frac{a'(1-\eta'^2)^2}{1+\eta'^2}, \quad r_0 = \frac{a(1-\eta^2)^2}{1+\eta^2}, \quad \alpha = \frac{r_0}{r_0'}, \quad D = \alpha\frac{\partial}{\partial\alpha},$$

$$\{1+\alpha^2 - 2\alpha\cos^2\tfrac{1}{2}I\cos(w-w')\}^{-\frac{1}{2}} = \Sigma a_i \cos i(w-w'), \ldots(3)$$

$$\tfrac{1}{2}\alpha\cos^2 I\,\{1+\alpha^2-2\alpha\cos^2\tfrac{1}{2}I\cos(w-w')\}^{-\frac{3}{2}} = \Sigma b_i\cos i(w-w'), \quad \ldots\ldots(4)$$

where $i = 0, \pm 1, \pm 2, \ldots$. Also, to the second order, $\eta = \frac{1}{2}e, \eta' = \frac{1}{2}e'$.

These definitions show at a glance the differences between the development given above and that of Leverrier, given by Tisserand*, in which $r_0' = a'$, $r_0 = a$, $D^j = \frac{\alpha^j}{j!}\left(\frac{\partial}{\partial\alpha}\right)^j$, $\cos^2\frac{1}{2}I = 1$ in (3), (4), a_i is replaced by $\frac{1}{2}a_i$ and b_i by $\frac{1}{2}b_i$, and there is no factor $\frac{1}{2}$ in the left-hand member of (4). When the necessary changes in notation have been made, the two developments will be found to agree with one another.

4·33. *The second term of R.*

The first term of the disturbing function, namely, m'/Δ, is the same for the action of either planet on the other, except as to the mass factor; the only condition actually used in the development is that accented letters shall refer to the outer planet. Omitting the mass factor, the second term is

$$-\frac{r\cos S}{r'^2} \text{ or } -\frac{r'\cos S}{r^2},$$

according as we are dealing with the action of the outer planet on the inner or that of the inner on the outer.

Expressed in rectangular coordinates these expressions are

$$-\frac{xx'+yy'+zz'}{r'^3}, \quad -\frac{xx'+yy'+zz'}{r^3}.$$

The equations for the elliptic motions of the planets are

$$\frac{d^2x'}{dt^2} = -\mu'\frac{x'}{r'^3}, \ldots, \ldots; \qquad \frac{d^2x}{dt^2} = -\mu\frac{x}{r^3}, \ldots, \ldots.$$

* *Méc. Cél.* vol. 1, p. 309.

If we put $\mu' = n'^2 a'^3$, $n'dt = dg'$ in the former and $\mu = n^2 a^3$, $n\,dt = dg$ in the latter, we may write these

$$-\frac{x'}{r'^3} = \frac{1}{a'^3}\frac{\partial^2 x'}{\partial g'^2}, \ldots, \ldots; \quad -\frac{x}{r^3} = \frac{1}{a^3}\frac{\partial^2 x}{\partial g^2}, \ldots, \ldots.$$

Since x', y', z' do not contain g and x, y, z do not contain g', the two cases of the second term of R may be written

$$\frac{1}{a'^3}\frac{\partial^2}{\partial g'^2}(rr'\cos S), \quad \frac{1}{a^3}\frac{\partial^2}{\partial g^2}(rr'\cos S),$$

which give a function $rr'\cos S$, symmetrical with respect to the two planets, to be developed.

These forms of the second term show that in the first case, there are no terms in the development in terms of the mean anomalies which contain the argument g only, and none in the second case containing g' only.

In neither case does the second term produce a constant part.

A quite general development in terms of the mean anomalies can be made. With the form 4·1 (3) for $\cos S$, it is evident that $rr'\cos S$ is a linear function of $r\cos v$, $r\sin v$ and therefore of $r\cos f$, $r\sin f$, and similarly of $r'\cos f'$, $r'\sin f'$, and these functions can be expressed in terms of the mean anomalies by the formulae 3·10 (5). An easy way to carry out the calculation is to use the second form of 4·1 (3) and write

$$rr'\cos S = A'.\,r\cos f - B'.\,r\sin f,$$

$$\begin{matrix} A' \\ B' \end{matrix} = \cos^2\tfrac{1}{2}I \begin{matrix} r'\cos \\ r'\sin \end{matrix}(\varpi - v') + \sin^2\tfrac{1}{2}I \begin{matrix} r'\cos \\ r'\sin \end{matrix}(\varpi + v' - 2\theta).$$

The series for A', B' may be computed by harmonic analysis, or in series from

$$A' = C_1.\,r'\cos f' + C_2.\,r'\sin f', \quad B' = -C_3.\,r'\sin f' + C_4.\,r'\cos f',$$

where

$$C_1, C_3 = \cos^2\tfrac{1}{2}I\cos(\varpi - \varpi') \pm \sin^2\tfrac{1}{2}I\cos(\varpi + \varpi' - 2\theta),$$

$$C_2, C_4 = \cos^2\tfrac{1}{2}I\sin(\varpi - \varpi') \mp \sin^2\tfrac{1}{2}I\sin(\varpi + \varpi' - 2\theta).$$

4·34. The expansion of $-r\cos S/r'^2$ can also be obtained from that of $1/\Delta$ in the following manner. If we put $\beta_s^{(i)}=0$ except for the cases $i=\pm 1$, $s=\frac{1}{2}$; $i=0$, $s=\frac{3}{2}$, and for these put the value $-\frac{1}{2}\alpha\kappa$, so that

$$a_1=a_{-1}=b_0=-\tfrac{1}{2}\alpha\kappa=-\tfrac{1}{2}\alpha\cos^2\tfrac{1}{2}I, \quad \ldots\ldots\ldots(1)$$

we find from 4·13 (4), 4·14 (1), that

$$R_1=-\alpha\kappa\cos(v-v'), \quad R_3=-\tfrac{1}{2}\alpha\kappa, \quad R_{2s}=0, \qquad s=\tfrac{5}{2},\tfrac{7}{2},\ldots.$$

With these values 4·13 (5) becomes

$$\frac{1}{\Delta_1}=-\alpha\kappa\cos(v-v')-\alpha\kappa\tan^2\tfrac{1}{2}I\cos(v+v'-2\theta)=-\alpha\cos S.$$

Finally if this be substituted in 4·8 (3) with $D=1$, we have

$$\frac{1}{\Delta}=\frac{1}{r_0'}\frac{\rho}{\rho'^2}(-\alpha\cos S)=-\frac{r\cos S}{r'^2}.$$

Thus the required expansion is obtained from that of $1/\Delta$ by putting $D=1$ and making the substitution (1).

To the second order, with

$$\alpha=\frac{r_0}{r_0'}=\frac{a}{a'}(1-3\eta^2+3\eta'^2),$$

we obtain from 4·32 (2), the result:

$$\begin{aligned}-\frac{r\cos S}{r'^2}=-\frac{a\cos^2\frac{1}{2}I}{a'^2}\{&(1-2\eta^2-2\eta'^2)\cos(g-g'+\varpi-\varpi')\\
&+\eta\cos(2g-g'+\varpi-\varpi')-3\eta\cos(g'-\varpi+\varpi')\\
&+4\eta'\cos(g-2g'+\varpi-\varpi')+4\eta\eta'\cos(2g-2g'+\varpi-\varpi')\\
&+\tfrac{3}{2}\eta^2\cos(3g-g'+\varpi-\varpi')+\tfrac{1}{2}\eta^2\cos(g+g'-\varpi+\varpi')\\
&+\tfrac{1}{2}\eta'^2\cos(g+g'+\varpi-\varpi')+\tfrac{27}{2}\eta'^2\cos(g-3g'+\varpi-\varpi')\\
&-12\eta\eta'\cos(2g'-\varpi+\varpi')\\
&+\tan^2\tfrac{1}{2}I\cos(g+g'+\varpi+\varpi'-2\theta)\}.\end{aligned}$$

The expansion of $-r'\cos S/r^2$ is evidently obtained by interchanging the accents.

CHAPTER V

CANONICAL AND ELLIPTIC VARIABLES

A. The Contact Transformation

5·1. *Canonical differential equations.*

Let the coordinates of a particle of mass m at any time t be x_1, x_2, x_3. If this particle moves under the force-function $U(x_1, x_2, x_3, t)$, the differential equations of its motion are

$$m\frac{d^2x_i}{dt^2}=\frac{\partial U}{\partial x_i}, \qquad i=1, 2, 3. \quad \ldots\ldots\ldots\ldots(1)$$

These are three equations of the second order. We shall now express them as six equations of the first order. This is accomplished by the introduction of the three new variables y_1, y_2, y_3, called momenta, defined by

$$y_i=m_i\frac{dx_i}{dt}, \qquad i=1, 2, 3. \ldots\ldots\ldots\ldots\ldots(2)$$

The kinetic energy is

$$T=\tfrac{1}{2}m\Sigma\left(\frac{dx_i}{dt}\right)^2$$
$$=\frac{1}{2m}\Sigma(y_i)^2. \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Differentiating (2) and substituting in (1) we have

$$\frac{dy_i}{dt}=\frac{\partial U}{\partial x_i}, \qquad i=1, 2, 3. \ldots\ldots\ldots\ldots\ldots(4)$$

From (2) and (3) we have

$$\frac{dx_i}{dt}=\frac{\partial T}{\partial y_i}, \qquad i=1, 2, 3 \quad \ldots\ldots\ldots\ldots\ldots(5)$$

Since U and T are independent of y_i and x_i respectively, (4) and (5) may be written, with $H=T-U$,

$$\frac{dx_i}{dt}=\frac{\partial H}{\partial y_i}, \quad \frac{dy_i}{dt}=-\frac{\partial H}{\partial x_i}, \qquad i=1, 2, 3. \ldots\ldots\ldots(6)$$

A set of differential equations in this form is said to be in the canonical or Hamiltonian form, and H is called the Hamiltonian function.

On account of the definitions of x_i, y_i as independent variables, the equations (5) may be expressed in the symbolic form,

$$\Sigma\,(dx_i\,.\,\delta y_i - dy_i\,.\,\delta x_i) = dt\,.\,\delta H, \quad \text{...........}(7)$$

where the δx_i, δy_i are arbitrary variations of the x_i, y_i, and δH is the consequent variation of H. The definitions of the symbols d, δ, introduced in this manner, will be made more precise in 5·3 below.

5·2. *The Contact transformation.*

Let x_i, y_i be any $2n$ variables which satisfy the canonical differential equations,

$$\frac{dx_i}{dt} = \frac{\partial H}{\partial y_i}, \quad \frac{dy_i}{dt} = -\frac{\partial H}{\partial x_i}, \qquad i = 1, 2, \ldots, n, \quad \text{......}(1)$$

where H is a function of x_i, y_i, t only: it does not contain any derivatives. A contact transformation is a change from the $2n$ variables x_i, y_i to $2n$ new variables, p_i, q_i, which shall satisfy equations of the same form, namely,

$$\frac{dp_i}{dt} = \frac{\partial H'}{\partial q_i}, \quad \frac{dq_i}{dt} = -\frac{\partial H'}{\partial p_i}, \quad \text{..............}(2)$$

where H' is related to H by an equation to be given below, and is expressed as a function of p_i, q_i, t.

According to a theorem of Jacobi, relations between the old and new variables which fulfil the conditions can be expressed by the implicit equations*

$$y_i = \frac{\partial S}{\partial x_i}, \quad p_i = \frac{\partial S}{\partial q_i}, \qquad S = \text{funct.}\, x_i, q_i, t. \quad \text{......}(3)$$

The *determining function* S must thus be expressed as a function of one set, either the x_i or the y_i, of the old variables, and one set of the new. It must be so chosen that it is possible,

* No confusion will be caused between this use of the letter S and that in the development of the disturbing function.

by means of the equations (3), to express the x_i, y_i in terms of the p_i, q_i, or vice versa. This possibility depends on the Jacobian,

$$\frac{\partial(x_1, \ldots, x_n, y_1, \ldots, y_n)}{\partial(p_1, \ldots, p_n, q_1, \ldots, q_n)}, \qquad \text{.................. (4)}$$

which must not vanish identically.

The literature connected with canonical equations and contact transformations is extensive and can be found from the usual sources of information. It may be mentioned, however, that while Hamilton appears to have first given the canonical forms of the equations of motion (*Brit. Ass. Report*, 1834, p. 513), Lagrange had given the equations for the variations of the elliptic elements in 1809 (*Mém. de l'Inst. de Paris*, p. 343) in this form. The theorem of Jacobi appeared in the *Comptes Rendus* for 1837, p. 61, the contact transformation having been introduced by Hamilton in 1828 (*Trans. Roy. Irish Acad.* vol. 15, p. 69).

Of the numerous applications of the theory of contact transformations we shall give only those which are necessary for the later developments in this volume. In particular, the proof of Jacobi's theorem, given in 5·3, does not indicate the process of discovery, but it has the advantage of showing immediately, not only the relation between H, H', S, but also the method which appears to be most useful in the search for new forms of canonical variables.

5·3. *Proof of the Jacobian transformation theorem.*

For the purposes of this proof and of later developments, it is desirable to define in more detail the meanings to be attached to the symbols d, δ in equations involving differentials.

The equations 5·2 (1) imply the existence of solutions of the form

$$x_i = x_i(t, a_1, \ldots, a_{2n}), \quad y_i = y_i(t, a_1, \ldots, a_{2n}),$$

and the relations 5·2 (3), the existence of functions,

$$p_i = p_i(t, a_1, \ldots, a_{2n}), \quad q_i = q_i(t, a_1, \ldots, a_{2n}),$$

which are solutions of 5·2 (3); in these expressions, $a_1, \ldots, a_{2n}$ are the arbitrary constants of the solution.

The symbol d attached to any function will always denote that when the function has been expressed in terms of t and the a_r, it is t alone which is varied, while the symbol δ implies

that any or all of the a_r are varied but that t is not changed. Thus when x_i, y_i are expressed in the forms just set down,

$$dx_i = \frac{\partial x_i}{\partial t} dt, \quad \delta x_i = \Sigma_r \frac{\partial x_i}{\partial a_r} \delta a_r, \qquad r = 1, 2, \ldots, 2n,$$

and thence, when S is expressed as a function of x_i, q_i, t,

$$dS = \Sigma \left(\frac{\partial S}{\partial x_i} dx_i + \frac{\partial S}{\partial q_i} dq_i \right) + \frac{\partial S}{\partial t} dt,$$

$$\delta S = \Sigma \left(\frac{\partial S}{\partial x_i} \delta x_i + \frac{\partial S}{\partial q_i} \delta q_i \right).$$

In these expressions for dS, δS, the meanings to be attached to dx_i, δx_i are those just given; similar meanings are to be attached to dq_i, δq_i.

Since the variations denoted by d, δ are independent, the commutative law, namely, that $\delta \,.\, d$ and $d \,.\, \delta$ acting on any function produce the same result, is satisfied.

The proof that the relations 5·2 (3) transform 5·2 (1) into 5·2 (2) follows.

Multiply 5·2 (1) by δy_i, $-\delta x_i$, respectively, and add for all values of i. We obtain

$$\Sigma \left(\frac{dx_i}{dt} \delta y_i - \frac{dy_i}{dt} \delta x_i \right) = \Sigma \left(\frac{\partial H}{\partial y_i} \delta y_i + \frac{\partial H}{\partial x_i} \delta x_i \right) = \delta H. \quad \ldots(1)$$

A similar process performed with 5·2 (3) gives

$$\Sigma \left(y_i \delta x_i + p_i \delta q_i \right) = \Sigma \left(\frac{\partial S}{\partial x_i} \delta x_i + \frac{\partial S}{\partial q_i} \delta q_i \right) = \delta S, \quad \ldots\ldots\ldots(2)$$

$$\Sigma \left(y_i dx_i + p_i dq_i \right) = \Sigma \left(\frac{\partial S}{\partial x_i} dx_i + \frac{\partial S}{\partial q_i} dq_i \right) = dS - \frac{\partial S}{\partial t} dt, \quad \ldots\ldots\ldots(3)$$

since S may contain t explicitly as well as through x_i, q_i.

Operate on (2) with d and on (3) with δ and subtract. Since the operators d, δ are commutative, we have $d\delta x_i = \delta dx_i$, $d\delta S = \delta dS$, etc., and all the terms in which both δ, d act on the same function disappear. We obtain

$$\Sigma \left(dy_i \,\delta x_i - dx_i \,\delta y_i \right) + \Sigma \left(dp_i \,\delta q_i - dq_i \,\delta p_i \right) = dt \,.\, \delta \left(\frac{\partial S}{\partial t} \right),$$

which is the same as

$$\Sigma\left(\frac{dy_i}{dt}\delta x_i - \frac{dx_i}{dt}\delta y_i\right) + \Sigma\left(\frac{dp_i}{dt}\delta q_i - \frac{dq_i}{dt}\delta p_i\right) = \delta\left(\frac{\partial S}{\partial t}\right).$$

The addition of (1) to this last equation gives

$$\Sigma\left(\frac{dp_i}{dt}\delta q_i - \frac{dq_i}{dt}\delta p_i\right) = \delta\left(H + \frac{\partial S}{\partial t}\right). \qquad \text{.........(4)}$$

Finally, if we define H' by means of the equation

$$H' = H + \frac{\partial S}{\partial t}, \qquad \text{.........(5)}$$

and suppose that H' has been expressed in terms of p_i, q_i, t, by means of the relations 5·2 (3), so that

$$\delta H' = \Sigma\left(\frac{\partial H'}{\partial p_i}\delta p_i + \frac{\partial H'}{\partial q_i}\delta q_i\right), \qquad \text{.........(6)}$$

the independence of δp_i, δq_i furnishes, through the equality of the members of (4), (6),

$$\frac{dp_i}{dt} = \frac{\partial H'}{\partial q_i}, \quad \frac{dq_i}{dt} = -\frac{\partial H'}{\partial p_i}. \qquad \text{.........(7)}$$

In the use of this transformation, it is important to remember that $\partial S/\partial t$ is found from the expression for S in 5·2 (3) and that H' is to be expressed in terms of p_i, q_i, t.

B. Jacobi's Partial Differential Equations

5·4. The transformation theorem just proved is a device for changing from one set of variables to another, the new variables depending on the choice of the determining function S. One such choice is the following.

Suppose that it is possible to find a form of S which will make the derivatives $\partial H'/\partial p_i$, $\partial H'/\partial q_i$ zero. (Since H' must necessarily be expressed in terms of p_i, q_i, t before these derivatives are formed, it follows that the derivatives will then be *identically* zero.) The equations 5·2 (2) show that dp_i/dt, dq_i/dt become zero, and hence that

$$p_i = \text{const.}, \quad q_i = \text{const.}$$

Jacobi showed that when S is so determined as to satisfy these conditions, it is a solution of a certain partial differential equation. In the next section, this equation will be found and those properties of it, which will be useful later, will be developed.

Notation. A semi-colon following a collective symbol x_i, where $i = 1, 2, \ldots, n$, will denote that any or all of the x_i may be present, thus

$$f(x_i;\ t) \text{ means } f(x_1, x_2, \ldots, x_n, t),$$

$$f(x_i;\ y_i;\ t) \text{ means } f(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n, t).$$

5·5. *The equation and its solutions.*

Amongst the values of S which will make p_i, q_i constant, we seek one which makes $H' = 0$, that is, one for which

$$H' = H + \frac{\partial S}{\partial t} = 0.$$

But H was originally a function of x_i, y_i, t, and, by 5·2 (3), $y_i = \partial S/\partial x_i$. Thus, we seek a value of S satisfying

$$H\left(x_i;\ \frac{\partial S}{\partial x_i};\ t\right) + \frac{\partial S}{\partial t} = 0. \quad \ldots\ldots\ldots\ldots\ldots(1)$$

Now the assumption concerning the form of S was that it should be expressed as a function of x_i, q_i, t, and the assumption is to be retained in (1). But, in the present case, the q_i are constants by hypothesis. Hence, in order to satisfy (1), we need an expression for S which contains x_i, t, and n arbitrary constants q_i. In other words, if we regard (1) as a partial differential equation with x_i, t as independent variables and with S as the dependent variable, we need a solution of the equation containing n arbitrary constants.

When such a solution has been obtained, all that is necessary is to interpret the relation 5·2 (3) in the language of the theory of differential equations, remembering that these now constitute $2n$ relations between the original variables x_i, y_i, the constants p_i, q_i, and t.

Theorem. A general solution of the equations

$$\frac{dx_i}{dt}=\frac{\partial H}{\partial y_i},\quad \frac{dy_i}{dt}=-\frac{\partial H}{\partial x_i},\qquad H=H(x_i;\ y_i;\ t)\quad \ldots(2)$$

is provided by the equations

$$y_i=\frac{\partial S}{\partial x_i},\quad p_i=\frac{\partial S}{\partial q_i},\quad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

where p_i, q_i are arbitrary constants, and S is an integral, containing n arbitrary constants q_i (exclusive of that additive to S) of the partial differential equation

$$H\left(x_i;\ \frac{\partial S}{\partial x_i};\ t\right)+\frac{\partial S}{\partial t}=0.$$

This type of integral is known as a *complete integral*, for the theory of which the reader is referred to treatises on first order partial differential equations. For our purposes, it is sufficient to state that the Jacobian 5·2 (4) must not vanish identically.

The constant additive to S plays no part because S appears in the differential equations and in the solution only through its derivatives, but its presence is theoretically necessary since there are $n+1$ independent variables in the partial differential equation.

It may be pointed out that the ordinary method for the solution of first order partial differential equations simply leads back to the canonical equations, so that nothing is gained by attempting to use it. In the applications to celestial mechanics, the form of the function S is usually set down from previous knowledge of the form of the solution, and the relations 5·2 (3) are then used in various ways; or, in particular cases, a form for S may be suggested by the equation 5·5 (1).

The set of arbitrary constants which appears in this way is known as a *canonical* set. It is evident that when the function S, containing half of them, has been obtained, the remaining half are chosen from the relations $p_i=\partial S/\partial q_i$. In general, therefore, only n of the $2n$ constants may be chosen arbitrarily. The practical demands of the perturbation problem limit the choice to very few types. See 5·12, 5·13, 5·14.

5·6. *The case where H is independent of t explicitly.*

A start may be made by assuming that

$$S = S_1 + Ct, \quad \text{..........................(1)}$$

so that the partial differential equation is reduced to

$$H\left(x_i;\ \frac{\partial S_1}{\partial x_i};\right) + C = 0, \quad \text{..................(2)}$$

which contains n independent variables only. If C be chosen as one of the canonical set of constants q_i, and t_0 be the corresponding constant derived from the equation $p_i = \partial S/\partial q_i$, we have

$$t_0 = \frac{\partial S_1}{\partial C} + t. \quad \text{..........................(3)}$$

Since S_1 does not contain t, it follows that the solution will contain t, t_0 only in the form $t - t_0$.

The equation (2) shows that $H =$ constant is an integral of the equations. This may be proved directly from the canonical equations 5·2 (1), by multiplying them by dy_i/dt, $-dx_i/dt$, and adding.

A similar procedure may be adopted when any one of the coordinates x_i is absent from H. It is evident that each absent coordinate permits the writing down of an integral of the equations.

5·7. *Application to the perturbation problem.*

The force-functions for the problem of three bodies which have been constructed in Chap. I, have usually been divided into the sum of two parts, the first of which, taken alone, gives elliptic motion. This division would lead to putting $U = U_0 + R$, where $U_0 = \mu/r$. It has been adopted because we can solve the equations completely when $R = 0$, and it has the added advantage that, since R usually contains a small factor, it constitutes a first approximation to the motion. These considerations, however, do not limit the applicability of the following method of procedure.

If U is replaced by $U_0 + R$, H is replaced by $T - U_0 - R$, or

by $H_0 - R$ if $H_0 = T - U_0$, so that the canonical equations will be written

$$\frac{dx_i}{dt} = \frac{\partial}{\partial y_i}(H_0 - R), \quad \frac{dy_i}{dt} = -\frac{\partial}{\partial x_i}(H_0 - R). \quad \text{......(1)}$$

Let us transform to a set of new variables p_i, q_i by means of the relations

$$y_i = \frac{\partial S}{\partial x_i}, \quad p_i = \frac{\partial S}{\partial q_i}, \quad \text{.....................(2)}$$

where S is a solution of the equation

$$H_0\left(x_i;\ \frac{\partial S}{\partial x_i};\ t\right) + \frac{\partial S}{\partial t} = 0. \quad \text{..................(3)}$$

The transformation theorem in 5·2 shows that the equations satisfied by the new variables are

$$\frac{dp_i}{dt} = -\frac{\partial R}{\partial q_i}, \quad \frac{dq_i}{dt} = \frac{\partial R}{\partial p_i}, \quad \text{..................(4)}$$

for in this case we have, by (3),

$$H' = H_0 - R + \frac{\partial S}{\partial t} = -R.$$

The interpretation of this result, usually adopted, is the following. If we solve the equations with $R = 0$ and obtain a canonical set of *constants* p_i, q_i, and consider these constants as variables when $R \neq 0$, the equations which they will satisfy are those numbered (4); hence the phrase, *Variation of Arbitrary Constants.* This latter point of view is useful in geometrical descriptions of the motion, but it sometimes leads to confusion and error if it is adopted in the analytical work. As a matter of fact, the equations with $R = 0$ are solved mainly in order to indicate the choice of the new variables, and are not used unless we know in advance that a solution can be obtained.

5·8. *Osculating orbits.*

Another geometrical interpretation of the results of 5·7 is of value in the determination of the orbit of a body from observations of its position. Let us suppose that the problem of finding

the solution of 5·7 (4) has been solved, so that the variables p_i, q_i are expressed as functions of t and arbitrary constants. If we put $t=t_0$ in this solution, where t_0 is some particular value of t, these variables become constants and thus constitute a solution of the equations 5·7 (4) when $R=0$. Hence at the instant $t=t_0$, the variables and the constants have the same value. But the coordinates and velocities are expressible in terms of t and p_i, q_i. It follows that the orbits with $R=0$, $R\neq 0$, intersect at $t=t_0$ and have the same velocities at the point of intersection; when this happens the two orbits are said to osculate at that point, and the ellipse described with $R=0$ is called the *osculating ellipse**.

If at the instant $t=t_0$, the disturbing forces which arise through R were suddenly annihilated, the body would thereafter proceed to move in the osculating ellipse. This constitutes another definition of this curve.

In the great majority of cases arising in the solar system, the forces due to R are small compared with those present when $R=0$, so that the osculating ellipse constitutes a good approximation to the orbit at times near $t=t_0$. In the case of a planet, the separation is small during a period of one revolution of the planet round the sun. Thus the osculating ellipse can be used to predict approximately the place of the body for some time before and after the instant $t=t_0$.

Ordinarily two coordinates, which give the angular position as seen from the earth, are observed; neither the distance nor the velocities are directly observed. From three such observations an osculating ellipse can in general be deduced. A position predicted for some other time in order to limit the area of search also needs only the two coordinates. There are six constants present in the osculating ellipse, and according to the mathematical theory of the approximate representation of a curve, considerable variations may be made in the six constants without altering the two needed functions of them very greatly. Thus the elements of an osculating orbit may be considerably in error and yet it may furnish a good search ephemeris for a considerable interval following its determination.

* Since the curvatures are not in general the same, the word 'osculating' is not used in the same sense as in the theory of curves.

C. Jacobian Method of Solution

5·9. *Solution of the case of elliptic motion by the Jacobian method.*

The force-function in this case is $m\mu/r$, and the canonical equations of motion become, after division by m,

$$\frac{dx_i}{dt} = \frac{\partial H_0}{\partial y_i}, \quad \frac{dy_i}{dt} = -\frac{\partial H_0}{\partial x_i},$$

where

$$H_0 = \tfrac{1}{2}(y_1^2 + y_2^2 + y_3^2) - \mu/r, \quad y_i = dx_i/dt.$$

The division by m is analytically equivalent to putting $m = 1$ in the formulae of 5·1.

According to 5·5, the equation satisfied by S is

$$\tfrac{1}{2}\left\{\left(\frac{\partial S}{\partial x_1}\right)^2 + \left(\frac{\partial S}{\partial x_2}\right)^2 + \left(\frac{\partial S}{\partial x_3}\right)^2\right\} - \frac{\mu}{(x_1^2 + x_2^2 + x_3^2)^{\frac{1}{2}}} + \frac{\partial S}{\partial t} = 0. \quad \text{......(1)}$$

Transform to the tri-polar coordinates r, $\frac{1}{2}\pi - L$, λ, so that L is the latitude, and λ the longitude of the projection of r on the plane of reference. The equation becomes

$$\tfrac{1}{2}\left\{\left(\frac{\partial S}{\partial r}\right)^2 + \left(\frac{\partial S}{r\,\partial L}\right)^2 + \left(\frac{\partial S}{r\cos L\,\partial\lambda}\right)^2\right\} - \frac{\mu}{r} + \frac{\partial S}{\partial t} = 0. \text{...(2)}$$

According to the preceding theory, we need an integral of (2) containing three arbitrary constants, exclusive of that additive to S. Since t, λ enter only through derivatives, it is convenient to put

$$S = -\alpha_1 t + \alpha_3\lambda + S_1, \quad \text{..................(3)}$$

where S_1 is independent of λ, t, and α_1, α_3 are constants. Inserting this value of S in (2) we obtain, after multiplication by $2r^2$ and rearrangement,

$$2\alpha_1 r^2 + 2\mu r - r^2\left(\frac{\partial S_1}{\partial r}\right)^2 = \left(\frac{\partial S_1}{\partial L}\right)^2 + \frac{\alpha_3^2}{\cos^2 L}. \quad \text{......(4)}$$

The form of this equation indicates that we can obtain a solution by putting $S_1 = S_2 + S_3$, where S_2 is a function of r only

and S_3 that of L only, if we equate each member to a constant α_2^2. This procedure gives

$$\left(\frac{\partial S_2}{\partial r}\right)^2 = 2\alpha_1 + \frac{2\mu}{r} - \frac{\alpha_2^2}{r^2}, \quad \left(\frac{\partial S_3}{\partial L}\right)^2 = \alpha_2^2 - \frac{\alpha_3^2}{\cos^2 L}.$$

The integration of these equations will leave two arbitrary constants at our disposal since the necessary three arbitrary constants $\alpha_1, \alpha_2, \alpha_3$ are already present. Let S_2 vanish for $r = r_1$, where r_1 is the smaller root of the equation

$$2\alpha_1 + \frac{2\mu}{r} - \frac{\alpha_2^2}{r^2} = 0, \quad \text{.....................(5)}$$

and let S_3 vanish when $L = 0$. In order that this value of r may be positive, it is necessary that both roots be positive; hence α_1 must be negative.

Inserting the values of S_2, S_3, S_1, thus obtained, in (3) we obtain a solution in the required form:

$$S = -\alpha_1 t + \alpha_3 \lambda + \int_{r_1}^{r} \left(2\alpha_1 + \frac{2\mu}{r} - \frac{\alpha_2^2}{r^2}\right)^{\frac{1}{2}} dr + \int_0^L \left(\alpha_2^2 - \frac{\alpha_3^2}{\cos^2 L}\right)^{\frac{1}{2}} dL. \quad \text{......(6)}$$

The next step is the deduction of the solution of the canonical equations by means of the relations $\beta_i = \partial S/\partial \alpha_i$, $y_i = \partial S/\partial x_i$, which would seem to demand a return to rectangular coordinates. However, we do not need the latter set since the former gives the necessary three relations between r, L, λ, t, and the relations between r, L, λ and x_1, x_2, x_3, are independent of α_1, α_2, α_3.

The derivatives of S with respect to the α_i are obtained without carrying out the quadratures in (6), by means of the formula

$$\frac{d}{d\alpha}\int_A^B f(x, \alpha)\, dx = \int_A^B \frac{\partial}{\partial \alpha} f(x, \alpha)\, dx + f(B, \alpha)\frac{\partial B}{\partial \alpha} - f(A, \alpha)\frac{\partial A}{\partial \alpha}.$$

The derivative with respect to α_1 gives

$$\beta_1 = -t + \int_{r_1}^{r} \frac{dr}{\left(2\alpha_1 + \frac{2\mu}{r} - \frac{\alpha_2^2}{r^2}\right)^{\frac{1}{2}}}, \quad \text{...........(7)}$$

since the coefficient of $\partial r_1/\partial \alpha$ vanishes on account of (5).

Similarly,

$$\beta_2 = -\int_{r_1}^{r} \frac{\alpha_2 dr}{r^2\left(2\alpha_1 + \frac{2\mu}{r} - \frac{\alpha_2^2}{r^2}\right)^{\frac{1}{2}}} + \int_0^L \frac{\alpha_2 dL}{\left(\alpha_2^2 - \frac{\alpha_3^2}{\cos^2 L}\right)^{\frac{1}{2}}}, \quad \ldots(8)$$

$$\beta_3 = \lambda - \int_0^L \frac{\alpha_3 dL}{\cos^2 L\left(\alpha_2^2 - \frac{\alpha_3^2}{\cos^2 L}\right)^{\frac{1}{2}}}. \quad \ldots\ldots\ldots\ldots(9)$$

The relations (7), (8), (9) are those needed, and the α_i, β_i constitute a set of canonical constants.

5·10. *Relations between the set α_i, β_i and a or n, e, i, ϵ, ϖ, θ.*

Since the integrand of 5·9 (7) must be real, the equation 5·9 (5) gives the maximum and minimum values of r. In 3·2 these are shown to be $a(1 \pm e)$. Hence from a well-known theorem connecting the roots and coefficients of a quadratic equation, $-2\mu/2\alpha_1 = 2a$, $-\alpha_2^2/2\alpha_1 = a^2(1-e^2)$, giving $\alpha_1 = -\mu/2a$, $\alpha_2 = \sqrt{\mu a(1-e^2)}$.

According to 5·9 (7), $-\beta_1$ is the value of t when $r = r_1 = a(1-e)$. But the mean anomaly $nt + \epsilon - \varpi$ is zero for this value of r. Hence $\beta_1 = (\epsilon - \varpi)/n$.

Equation 5·9 (9) shows that β_3 is the value of λ when $L = 0$, that is, when the body is in the plane of reference. Hence $\beta_3 = \theta$, the longitude of the node.

Since the integrand of 5·9 (9) must be real, the maximum and minimum values of $\cos L$ are $\pm \alpha_3/\alpha_2$. But the maximum and minimum values of L are $\pm i$, where i is the inclination of the plane of the orbit to the plane of reference. Hence $\alpha_3 = \alpha_2 \cos i$.

Finally, if we put $\alpha_3 = \alpha_2 \cos i$ in the last term of 5·9 (3) it becomes

$$\int_0^L \frac{\cos L\, dL}{(\cos^2 L - \cos^2 i)^{\frac{1}{2}}} = \int_0^L \frac{d(\sin L)}{(\sin^2 i - \sin^2 L)^{\frac{1}{2}}} = \sin^{-1}\left(\frac{\sin L}{\sin i}\right).$$

But if ν be the hypotenuse of the right-angled spherical triangle in which L is the side opposite to the angle i, we have $\sin L = \sin i \sin \nu$. Thus the above integral is ν and the equation

5·9 (8) shows that β_2 is the value of ν when $r = r_1$. Since ν is then the angle between the apse and the node, we have $\beta_2 = \varpi - \theta$.

Collecting these results, we obtain the system of canonical elements

$$\left.\begin{array}{ll} \alpha_1 = -\mu/2a, & \beta_1 = (\epsilon - \varpi)/n, \\ \alpha_2 = \sqrt{\mu a (1 - e^2)}, & \beta_2 = \varpi - \theta, \\ \alpha_3 = \cos i \sqrt{\mu a (1 - e^2)}, & \beta_3 = \theta. \end{array}\right\} \quad \text{......(1)}$$

Hence, according to the principles set forth above, the equations satisfied by the α_i, β_i when $R \neq 0$, are

$$\Sigma\, (d\alpha_i\, \delta\beta_i - d\beta_i\, \delta\alpha_i) = dt \,.\, \delta R, \quad \text{............(2)}$$

or, written *in extenso*,

$$\frac{d\alpha_i}{dt} = \frac{\partial R}{\partial \beta_i}, \quad \frac{d\beta_i}{dt} = -\frac{\partial R}{\partial \alpha_i}, \qquad i = 1, 2, 3. \quad \text{.........(3)}$$

5·11. The fact that 5·9 (7), (8), (9) give elliptic motion may be deduced in the following manner.

If we put $r = a - ae\cos X$ in (7), and insert the values of the constants found above, we obtain, after integration,

$$\frac{\epsilon - \varpi}{n} = -t + \left(\frac{a^3}{\mu}\right)^{\frac{1}{2}} (X - e \sin X),$$

which, with the aid of the relation $a^3 n^2 = \mu$, is seen to be the equation connecting the mean and eccentric anomalies.

Equation 5·9 (8), with the same substitution for r, becomes

$$\varpi - \theta = -\int_0^X \frac{dX}{1 - e\cos X} + \nu;$$

since $\nu = v - \theta$, $f = v - \varpi$, this equation gives the relation between the true and eccentric anomalies.

Finally, the substitution $\sin L = \sin i \sin \nu$ in the integral of (9) gives

$$\theta = \lambda - \tan^{-1}(\cos i \tan \nu);$$

since $\lambda - \theta$ is the side adjacent to the angle i in the right-angled triangle to which reference is made in the text, this equation merely constitutes a well-known geometrical relation.

A logical procedure requires the proof that the motion is elliptic to precede the identification of the constants. But as the objective in view is the discovery of a set of canonical constants only, we assumed that the nature of the motion had been found by the easier method of Chap. III. (Cf. 3·2.)

D. Other Canonical and Non-canonical Sets

5·12. *Delaunay's canonical elements.*

Changes from one canonical set of elements α_i, β_i to another such set can sometimes be carried through easily by the Jacobian transformation theorem proved in 5·3 if we use as the determining function

$$S = \Sigma \alpha_i \beta_i, \quad \text{......................(1)}$$

where the β_i (or the α_i) are expressed in terms of half the variables of the new set.

Let us take as three of the variables of a new set, l, g, h, defined by

$$g = \varpi - \theta = \beta_2, \quad h = \theta = \beta_3,$$

$$l = nt + \epsilon - \varpi = n(t + \beta_1) = \left(\frac{\mu}{a^3}\right)^{\frac{1}{2}}(t + \beta_1) = \frac{(-2\alpha_1)^{\frac{3}{2}}}{\mu}(t + \beta_1),$$

so that $\qquad \beta_1 = \mu(-2\alpha_1)^{-\frac{3}{2}} - t.$

Hence S, expressed in the required form, is

$$S = \mu(-2\alpha_1)^{-\frac{1}{2}} l - \alpha_1 t + \alpha_2 g + \alpha_3 h. \quad \text{.........(2)}$$

If the other three new variables be L, G, H, we have, by 5·2 (3),

$$L = \frac{\partial S}{\partial l} = \mu(-2\alpha_1)^{-\frac{1}{2}} = \sqrt{\mu a}, \quad G = \frac{\partial S}{\partial g} = \alpha_2, \quad H = \frac{\partial S}{\partial h} = \alpha_3. \quad \text{......(3)}$$

The remaining three equations are automatically satisfied. Also

$$H' = H + \frac{\partial S}{\partial t} = R - \alpha_1 = R + \frac{\mu^2}{2L^2} \quad \text{............(4)}$$

Thus this set, which is that used by Delaunay, satisfies the equation

$$dL\,\delta l - dl\,\delta L + dG\,\delta g - dg\,\delta G + dH\,\delta h - dh\,\delta H = dt\,\delta\left(R + \frac{\mu^2}{2L^2}\right), \quad \text{......(5)}$$

where

$$\left.\begin{aligned} L &= \sqrt{\mu a}, & l &= nt + \epsilon - \varpi, \\ G &= \sqrt{\mu a(1 - e^2)}, & g &= \varpi - \theta, \\ H &= \cos i\,\sqrt{\mu a(1 - e^2)}, & h &= \theta. \end{aligned}\right\} \quad \text{......(6)}$$

The variables in each of the two groups are homogeneous; L, G, H are angular momenta or, since the mass acted on has been divided out, areal velocities. If we put $\mu = n^2 a^3$, the common factor $\sqrt{a\mu}$ becomes na^2.

5·13. *The modified Delaunay set.*

If we use the determining function $S = Ll + Gg + Hh$, in the form

$$S = L(l + g + h) + (G - L)(g + h) + (H - G)h,$$

the equations of transformation show that L, $G - L$, $H - G$ and $l + g + h$, $g + h$, h form a canonical set. We shall denote them by c_i, w_i, so that

$$\left.\begin{aligned} c_1 &= \sqrt{\mu a}, & w_1 &= nt + \epsilon = \text{mean longitude}, \\ c_2 &= \sqrt{\mu a}\,(\sqrt{1 - e^2} - 1), & w_2 &= \varpi = \text{long. of apse}, \\ c_3 &= \sqrt{\mu a\,(1 - e^2)}\,(\cos i - 1), & w_3 &= \theta\ = \text{long. of node}. \end{aligned}\right\} \quad \text{......(1)}$$

The Hamiltonian function is unaltered and is equal to $R + \mu^2/2c_1^2$.

When expansion is made in powers of e, i, the element c_2 is divisible by e^2 and c_3 by i^2—properties which make this set useful in planetary problems.

5·14. *A set given by Poincaré**.

This set is c_1, p_2, p_3; w, q_2, q_3, defined by

$$\begin{aligned} c_1 &= \sqrt{\mu a}, & w &= nt + \epsilon, \\ p_2 &= \sqrt{-2c_2}\sin\varpi, & q_2 &= \sqrt{-2c_2}\cos\varpi, \\ p_3 &= \sqrt{-2c_3}\sin\theta, & q_3 &= \sqrt{-2c_3}\cos\theta, \end{aligned}$$

where c_2, c_3 are defined in 5·13 (1). That it is canonical can be tested by showing that

$$dp_2\,\delta q_2 - dq_2\,\delta p_2 = dc_2\,\delta w_2 - dw_2\,\delta c_2,$$

with a similar equation for the suffix 3.

This is a particular case of a general theorem† which states

* *Les Nouv. Mét. de la Méc. Cél.* vol. 1, p. 30.

† C. A. Shook, *l.c.* in 5·5.

that if two variables p_2, q_2 are related to two canonical variables c_2, w_2 in such a manner that the Jacobian,

$$\frac{\partial (p_2, q_2)}{\partial (c_2, w_2)} \equiv 1,$$

p_2, q_2 are also canonical variables.

The Hamiltonian function is $R + \mu^2/2c_1^2$, as before.

This set is useful because the approximate values of p_2, q_2 are

$$\sqrt{\mu a} \,.\, e \sin \varpi, \quad \sqrt{\mu a} \,.\, e \cos \varpi,$$

when e^3 is neglected, and those of p_3, q_3 are

$$\sqrt{\mu a (1 - e^2)} \,.\, i \sin \theta, \quad \sqrt{\mu a (1 - e^2)} \,.\, i \cos \theta,$$

when i^3 is neglected, so that the disturbing function is developable in powers of p_2, p_3, q_2, q_3, such a development replacing powers of e, i, and cosines and sines of multiples of ϖ, θ. The possibility of such a development depends on the association of powers of e with multiples of ϖ, and of powers of i with multiples of θ when the angles are expressed in terms of ϖ, ϖ', θ and the mean longitudes. See 4·15.

5·15. *The non-canonical set a, e_1, Γ_1, w, ϖ, θ, where $w = nt + \epsilon$,*

$$e_1 = \sqrt{2 (1 - \sqrt{1 - e^2})} = e + \tfrac{1}{8} e^3 \ldots, \quad \Gamma_1 = (1 - \cos i) \sqrt{1 - e^2}. \quad \ldots\ldots(1)$$

The disturbing function is not usually expressed in terms of the preceding canonical sets of elements. The angular variables are present explicitly, but the remaining elements are mixtures of a, e, i which are present explicitly. The same is true of the coordinates. For calculation, it is usually easier to adopt elements which are more directly related to those which are explicitly present in the ordinary developments of R and of the coordinates. The equations satisfied by such elements are not canonical.

The definitions of e_1, Γ_1 given in (1) show that these variables are related to c_1, c_2, c_3 by the equations

$$c_1 = \sqrt{\mu a}, \quad c_2 = -\tfrac{1}{2} e_1^2 \sqrt{\mu a}, \quad c_3 = -\Gamma_1 \sqrt{\mu a}. \quad \ldots\ldots(2)$$

The variables w, ϖ, θ are the same as those denoted in 5·13 (1) by w_1, w_2, w_3.

The transformation to the new variables is most easily effected by forming their variations from (2) and substituting them in the left-hand member of the equation,

$$\Sigma dc_i \delta w_i - \Sigma dw_i \delta c_i = dt\,\delta\,(R + \mu^2/2c_1^2). \quad \ldots\ldots\ldots(3)$$

The process is quite straightforward. After the variations have been formed, it is convenient to use the equation $\mu = n^2 a^3$ by putting $\sqrt{\mu a} = \mu/na$, $\sqrt{\mu/a} = \mu/na^2$. After rearrangement, we obtain for the left-hand member of (3) the expression

$$\begin{aligned}(-\tfrac{1}{2}dw + \tfrac{1}{4}e_1^2 d\varpi + \tfrac{1}{2}\Gamma_1 d\theta)\,\delta a + ae_1 d\varpi\,\delta e_1 + a\,d\theta\,\delta\Gamma_1 \\ + \tfrac{1}{2}da\,\delta w + (-\tfrac{1}{4}e_1^2 da - ae_1 de_1)\,\delta\varpi + (-\tfrac{1}{2}\Gamma_1 da - a\,d\Gamma_1)\,\delta\theta\end{aligned}$$

multiplied by μ/na^2.

Since $\mu^2/2c_1^2 = \mu/2a$, we have, if R be supposed expressed in terms of the new elements,

$$\begin{aligned}\delta\left(R + \frac{\mu^2}{2c_1^2}\right) = \left(\frac{\partial R}{\partial a} - \frac{\mu}{2a^2}\right)\delta a + \frac{\partial R}{\partial e_1}\delta e_1 + \frac{\partial R}{\partial \Gamma_1}\delta\Gamma_1 \\ + \frac{\partial R}{\partial w}\delta w + \frac{\partial R}{\partial \varpi}\delta\varpi + \frac{\partial R}{\partial \theta}\delta\theta,\end{aligned}$$

which is substituted in the right-hand member of (3).

Since the variations of the new elements are, like those of the old elements, independent, we can equate their coefficients on the two sides of the equation. On solving the six equations thus obtained so as to isolate da/dt, de_1/dt, ..., we obtain

$$\left.\begin{aligned}&\frac{1}{a}\frac{da}{dt} = -\frac{2}{3}\frac{1}{n}\frac{dn}{dt} = \frac{2na}{\mu}\frac{\partial R}{\partial w}, \\ &\frac{dw}{dt} = n - \frac{2na^2}{\mu}\frac{\partial R}{\partial a} + \frac{nae_1}{2\mu}\frac{\partial R}{\partial e_1} + \frac{na\Gamma_1}{\mu}\frac{\partial R}{\partial \Gamma_1}, \\ &e_1\frac{de_1}{dt} = -\frac{na}{\mu}\frac{\partial R}{\partial \varpi} - \frac{nae_1^2}{2\mu}\frac{\partial R}{\partial w}, \quad \frac{d\varpi}{dt} = \frac{na}{\mu e_1}\frac{\partial R}{\partial e_1}, \\ &\frac{d\Gamma_1}{dt} = -\frac{na}{\mu}\frac{\partial R}{\partial \theta} - \frac{na\Gamma_1}{\mu}\frac{\partial R}{\partial w}, \quad \frac{d\theta}{dt} = \frac{na}{\mu}\frac{\partial R}{\partial \Gamma_1}.\end{aligned}\right\} \ldots(4)$$

The objective in this transformation is the isolation of derivatives with respect to a, so that the operator D, which plays so large a part in the development of the disturbing function, may

act only on those portions which are specifically set forth in Chap. IV.

Since R has the dimensions, mass divided by distance, and n has the dimension of the inverse of a time, the factor μ/na^2 reduces all the equations to relations between ratios.

The right-hand members of all the equations except the second contain the small factor present in R. The value of n found from the first equation is to be substituted for the term n of the second equation before the latter is integrated.

The relations

$$e = e_1(1 - \tfrac{1}{4}e_1^2)^{\frac{1}{2}}, \quad 2\eta = e_1(1 - \tfrac{1}{4}e_1^2)^{-\frac{1}{2}}, \quad \ldots\ldots\ldots(5)$$

deduced from (1) and from $e = 2\eta/(1+\eta^2)$, permit R to be easily expressed in terms of e_1; $e_1 - e$, $2\eta - e_1$ are approximately $\frac{1}{8}e^3$, so that in many problems it would be possible to neglect these differences. It is recalled that e, and therefore e_1, is present in α as defined in the developments of Chap. IV; it is also present in Γ_1. Hence if R be developed as in that chapter, we have to substitute for $\partial R/\partial e_1$ in the equations (4), the value derived from

$$\left(\frac{\partial R}{\partial e} + \frac{\partial R}{\partial \Gamma}\frac{\partial \Gamma}{\partial e} + DR \,.\, \frac{1}{\alpha}\frac{\partial \alpha}{\partial e}\right)\frac{de}{de_1},$$

when $\partial\Gamma/\partial e$ is formed from $\Gamma = \Gamma_1(1-e^2)^{-\frac{1}{2}}$ and de/de_1 from (5). For $\partial R/\partial \Gamma_1$ we have $(\partial R/\partial \Gamma)(1-e^2)^{-\frac{1}{2}}$.

5·16. *The non-canonical set a, e, Γ or i, w or ϵ, ϖ, θ.*

This set is deduced from 5·15 (4) by means of the relations 5·15 (1) or 5·15 (5), through the usual process of a change of variables from e_1 to e and Γ_1 to Γ. The result is the following set of equations

$$\frac{1}{a}\frac{da}{dt} = -\frac{2}{3}\frac{1}{n}\frac{dn}{dt} = \frac{2na}{\mu}\frac{\partial R}{\partial w},$$

$$\frac{dw}{dt} = n - \frac{2na}{\mu}a\frac{\partial R}{\partial a} + \frac{e\sqrt{1-e^2}}{1+\sqrt{1-e^2}}\frac{na}{\mu}\frac{\partial R}{\partial e} + \frac{\Gamma}{\sqrt{1-e^2}}\frac{na}{\mu}\frac{\partial R}{\partial \Gamma},$$

$$\frac{de}{dt} = -\sqrt{1-e^2}\,\frac{na}{\mu e}\frac{\partial R}{\partial \varpi} - \frac{e\sqrt{1-e^2}}{1+\sqrt{1-e^2}}\frac{na}{\mu}\frac{\partial R}{\partial w},$$

$$\frac{d\varpi}{dt} = \sqrt{1-e^2}\,\frac{na}{\mu e}\frac{\partial R}{\partial e} + \frac{\Gamma}{\sqrt{1-e^2}}\frac{na}{\mu}\frac{\partial R}{\partial \Gamma},$$

$$\frac{d\Gamma}{dt} = -\sqrt{1-e^2}\,\frac{na}{\mu}\frac{\partial R}{\partial \theta} - \frac{\Gamma}{\sqrt{1-e^2}}\frac{na}{\mu}\left(\frac{\partial R}{\partial \epsilon} + \frac{\partial R}{\partial \varpi}\right),$$

$$\frac{d\theta}{dt} = \frac{1}{\sqrt{1-e^2}}\frac{na}{\mu}\frac{\partial R}{\partial \Gamma}.$$

This set is slightly more convenient than the older form in which ϵ_1, i replace w, Γ respectively.

The definition of ϵ_1 is usually given by means of the equation

$$w = nt + \epsilon = \int n\,dt + \epsilon_1,$$

so that $d\epsilon_1/dt = dw/dt - n$. And as ϵ occurs in R and in the coordinates only in the combination $nt+\epsilon$, we have $\partial R/\partial w = \partial R/\partial \epsilon$. Nothing is gained by the substitution of ϵ_1 for w, except perhaps a separation, in the case of the long-period terms, of those portions which have the square of the small divisor as a factor, from those which have the first power of this divisor as a factor.

The substitution of $\Gamma = 1 - \cos i$ for i is advantageous because i occurs in the disturbing function only in the form $\cos i$, or rather $\cos I$ when the orbital plane of the disturbing planet is taken as the plane of reference. The older form can be at once deduced by means of the relations,

$$\frac{d\Gamma}{dt} = \sin i\,\frac{di}{dt}, \quad \Gamma\frac{\partial R}{\partial \Gamma} = \tan\tfrac{1}{2}i\,\frac{\partial R}{\partial i}.$$

5·17. *The case of attraction proportional to the distance from a fixed origin.*

An example of such a gravitational force is that on any one of a spherical arrangement of particles with a mass-density which is uniform throughout the sphere. Since such an arrangement cannot be exactly maintained under the Newtonian laws of force and gravitation, the force and resulting motion must be considered as approximate only. Another similar example is that of an arrangement of particles in an ellipsoid of revolution

with uniform mass-density: the force on a particle in the equatorial plane varies directly as the distance from the centre.

In the spherical case, the force is μmr, where m is the mass of the particle, r its distance from the origin and μ is $4\pi/3$ times the density multiplied by the gravitational constant. In the ellipsoidal case, μ depends also on the eccentricity of the ellipsoid.

The force-function for these ideal cases is $-\frac{1}{2}\mu mr^2$. Let mR denote the force-function for the remaining forces which act on m. If x_1, x_2, x_3 be the rectangular coordinates of m referred to fixed axes through the centre of the system, the equations of motion will be

$$\frac{d^2x_i}{dt^2} = \frac{\partial}{\partial x_i}(-\tfrac{1}{2}\mu r^2 + R), \qquad i = 1, 2, 3,$$

with $r^2 = x_1^2 + x_2^2 + x_3^2$.

When $R = 0$, each coordinate describes a harmonic motion with period $2\pi/n$, where $n^2 = \mu$, and the orbit is an ellipse whose centre is at the origin.

As before, the elements of this ellipse may be considered as new variables for the examination of the case $R \neq 0$. A canonical set, with Hamiltonian function R, in which μ is replaced by n^2, is the following:

$$L = \frac{a^2}{n}(1 - \tfrac{1}{2}e^2), \quad G = \frac{a^2}{n}(1 - e^2)^{\frac{1}{2}}, \quad H = \frac{a^2}{n}(1 - e^2)^{\frac{1}{2}} \cos i,$$

$$l_0 = \epsilon - \varpi, \qquad g = \varpi - \theta, \qquad h = \theta.$$

If l_0 be replaced by $l = nt + \epsilon - \varpi$, the only change needed is the replacement of R by $R - nL$ as the Hamiltonian function.

The proof is left to the reader. A modification similar to that of 5·13 may also be made.

CHAPTER VI

SOLUTION OF CANONICAL EQUATIONS

6·1. The main object of this chapter is the development of methods of solution for the types of canonical equations which have been obtained in the previous chapter. All the methods depend fundamentally on the assumption that the variables differ from constants by amounts which have as factor the ratio of the disturbing mass to that of the primary, and therefore that the variables may be developed in powers of this ratio. As long as we confine ourselves to the first power of this ratio there is little choice between the various methods; substantially, they are equivalent. It is when we need to take into account higher powers of the ratio that differences appear, mainly in the amount of calculation which is necessary to secure the desired accuracy.

The fundamental idea of the method of Delaunay, namely a change of variables such that the new variables are more nearly constant than the old ones, is used throughout. But the application of the idea is different from that which Delaunay made to the solution of the satellite problem, where the changes of variables were very numerous*. Here it is shown that one change of variables is, in general, sufficient for the solution of the majority of planetary problems provided the new variables are suitably chosen. Much of the discussion in the chapter hinges on the amount of labour which the development and solution of the equations for the new variables requires.

6·2. *Elliptic elements or variables.*

In Chap. v it has been shown that the equations of motion can be put into the canonical form, and methods of changing from the coordinates and velocities to other sets of variables are developed. Particular sets of new variables which are con-

* See for example, Tisserand, vol. 3, Chaps. XI, XII; E. W. Brown, *Lunar Theory*, Chap. IX.

nected with motion in an ellipse have been chosen and the equations satisfied by these new variables have been given in canonical form; other sets are given in non-canonical form. Various points of view of these new variables, usually called the elliptic elements, are given. It was, however, pointed out that for the purposes of mathematical development, they should be regarded merely as a set of new variables which are connected with the coordinates and velocities by a set of equations, the latter remaining the same for all values of the variables.

These new variables have the property of becoming constants or linear functions of t when $R=0$. In one set, they are all constants; in the other sets, one of them is a linear function of t.

The canonical set denoted by c_i, w_i and developed in 5·13, will be used in this chapter. Only slight changes, easily made, are needed if the set $c_1, p_2, p_3, w_1, q_2, q_3$, given in 5·14, be used. Since R is not in general developed in terms of canonical elements, it is shown how the work may be so adapted that the developments of R given in Chap. IV may be used.

6·3. *The disturbing function.*

In Chap. IV, the disturbing function has been expanded into a sum of periodic terms on the assumption that the motion of each planet is elliptic. This restriction can now be removed, so far as the disturbed planet is concerned. Since the development consisted in the replacement of the coordinates by their values in terms of the time and the elliptic elements, the relations between them being those referred to in 6·2, the development is unchanged if we consider these elements as variables.

The relations between the coordinates and velocities and the new variables c_i, w_i are independent of t explicitly; in elliptic motion t is implicitly present in w_1 only. Thus the only way in which t is explicitly present in R is through the coordinates of the disturbing planet, and it appears actually only through $w_1' = n't + \epsilon'$ or $g' = n't + \epsilon' - \varpi'$. If the motion of the disturbing planet is not elliptic, its deviations from elliptic motion can

be expressed as variations of its elements, so that the given development of R can still be utilised. The necessary modifications of the solution are not difficult; they are exhibited in one particular case (6·19 below).

The possibility of expanding R in power series when the canonical elements c_i, w_i are used has to be considered. Since $c_1 = \sqrt{\mu a}$, it is evident that c_1 can replace a without difficulty. Also, by 5·15 (2), we have

$$e_1 = \left(-\frac{2c_2}{c_1}\right)^{\frac{1}{2}}, \quad \Gamma_1 = \left(-\frac{c_3}{c_1}\right)^{\frac{1}{2}},$$

and since, by 5·15 (1) or 5·15 (5), we can replace e by e_1 and Γ by Γ_1 in the developments, it follows that developments in powers of e, Γ are replaceable by developments in powers of $(-2c_2/c_1)^{\frac{1}{2}}$, $(-c_3/c_1)^{\frac{1}{2}}$. Derivatives with respect to c_2, c_3 will implicitly involve the presence of negative powers of e, Γ, and it will be necessary to show that these negative powers disappear from the transformed disturbing function. The difficulty does not arise when the variables c_1, p_2, p_3, w_1, q_2, q_3 are used, because R and the coordinates are expansible in positive integral powers of p_2, p_3, q_2, q_3 (5·14).

We have, approximately, $p_2 = e \sin \varpi$, $q_2 = e \cos \varpi$; suppose that e_0, ϖ_0 are the undisturbed constant values of e, ϖ. We then get

$$e \sin \varpi = e_0 \sin \varpi_0 + \text{perturbations},$$

$$e \cos \varpi = e_0 \cos \varpi_0 + \text{perturbations}.$$

While the perturbations vanish with the disturbing mass, *they do not in general vanish with* e_0. Hence, even if the observed eccentricity of the disturbed body is so small as to be negligible, we cannot assume that e is negligible in finding its perturbations by the method of the variation of the elements. Thus, while geometrical descriptions of the motion are simple in terms of e, ϖ, the analytical development fundamentally requires the use of $e \sin \varpi$, $e \cos \varpi$, as convenient variables. The general control of the work is, however, much easier with the use of the former than with that of the latter variables. In certain cases, particularly

those in which e_0 is very small, and also in the discussion of the secular perturbations, it is advisable to use the variables p_2, q_2 rather than c_2, w_2. Similar remarks apply to p_3, q_3 and to Γ, θ, but the problem for these latter variables is less difficult because R is expansible in positive integral powers of Γ and therefore of c_3.

The solution of the problem of the apparent presence of negative powers of e is given in the theorems of 6·4, 6·15 below.

6·4. *D'Alembert series.*

The association of powers of e with multiples of ϖ, of powers of e' with multiples of ϖ', of powers of $\sqrt{\Gamma}$ with multiples of θ, and of powers of α with multiples of $w - w'$, which have been pointed out in the development of R (4·15), and in certain of the developments of Chap. III (3·3), are so useful that it was found convenient to give a designation to such series: we have named them *d'Alembert series**.

Certain of the expansions of Chaps. II, III, have been seen to be of this type. Certain other series are easily related to it. Thus, $\sin jf$, $\cos jf$ are not d'Alembert series with respect to the coefficient e and the angle g, but $\sin j(f-g)$, $\cos j(f-g)$ are.

It is evident that if we have a d'Alembert series with respect to a coefficient A and an angle x, the series is formally expansible in powers of $A \sin x$, $A \cos x$, and in powers of $A \exp. x\sqrt{-1}$, $A \exp. -x\sqrt{-1}$. A general property of these series is given in the following theorem.

If f, f' are d'Alembert series with respect to the coefficient A and the angle x, then the Jacobian,

$$\frac{\partial(f, f')}{\partial(A^2, x)} = \frac{1}{2A}\left(\frac{\partial f}{\partial A}\frac{\partial f'}{\partial x} - \frac{\partial f}{\partial x}\frac{\partial f'}{\partial A}\right),$$

is also a d'Alembert series with respect to A, x.

Let

$$y = A \exp. x\sqrt{-1}, \quad z = A \exp. -x\sqrt{-1}, \quad yz = A^2.$$

* It appears that d'Alembert was the first to notice this property of the disturbing function with respect to the eccentricities and longitudes of perihelia, in his memoir, "Recherches sur différens Points importans du Système du Monde," *Mém. Paris Acad. Sc.* 1754.

The transformation of the Jacobian from the variables A, x to the variables y, z gives

$$\frac{\sqrt{-1}}{2A}\left(\frac{y}{A}\frac{\partial f}{\partial y}+\frac{z}{A}\frac{\partial f}{\partial z}\right)\left(y\frac{\partial f'}{\partial y}-z\frac{\partial f'}{\partial z}\right)$$
$$-\frac{\sqrt{-1}}{A}\left(y\frac{\partial f}{\partial y}-z\frac{\partial f}{\partial z}\right)\left(\frac{y}{A}\frac{\partial f'}{\partial y}+\frac{z}{A}\frac{\partial f'}{\partial z}\right)$$
$$=\sqrt{-1}\left(\frac{\partial f}{\partial z}\frac{\partial f'}{\partial y}-\frac{\partial f}{\partial y}\frac{\partial f'}{\partial z}\right),$$

since $yz = A^2$. The definitions of f, f' make them expansible in powers of y, z and therefore their derivatives have the same property. The Jacobian is therefore a d'Alembert series.

Thus while the separate terms of the original Jacobian are not d'Alembert series, the divisor A disappears from the Jacobian when it is expressed in terms of A, x. This is the property we really require to know.

6·5. *Other properties of R.*

It is useful to recall certain of these.

(a) It is homogeneous and of degree -1 with respect to length. The variables c_i are homogeneous and of dimensions (mass)$^{\frac{1}{2}}$ (length)$^{\frac{1}{2}}$ or, with the unit of mass actually used, of dimensions (time)$^{-1}$ (length)^2.

(b) It depends only on relative coordinates and is therefore independent of the origin of measurement of the angles. The anomalies are by definition independent of such origin. When R is expressed in terms of g, g', ϖ, ϖ', θ, the only forms in which the latter three angles are present are their differences, usually expressed by $\varpi - \varpi'$, $\varpi + \varpi' - 2\theta$.

(c) It is a function of the coordinates only and is independent of the velocities. This property was utilised in forming the equations of motion. One result is the equation

$$\frac{d}{dt}\left(R+\frac{\mu^2}{2c_1^2}\right)=\frac{\partial}{\partial t}\left(R+\frac{\mu^2}{2c_1^2}\right)=\frac{\partial R}{\partial t}, \quad \text{.........(1)}$$

which is a consequence of the equation

$$\Sigma_i \left(\frac{\partial R}{\partial c_i}\frac{dc_i}{dt} + \frac{\partial R}{\partial w_i}\frac{dw_i}{dt}\right) = n\frac{\partial R}{\partial w_1}, \quad \ldots\ldots\ldots\ldots(2)$$

a result which is deducible from the canonical equations in 5·13, with the help of the relation $n = \mu/c_1^3$.

The relation (1) may also be written

$$\frac{dR}{dt} = \frac{\partial R}{\partial t} + n\frac{\partial R}{\partial w_1}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

by the use of the canonical equation for c_1.

When the two bodies are supposed to move in fixed ellipses, the relation (3) is evidently true, since t occurs only through g', g; the term $\partial R/\partial t$ enters through g' only and $\partial R/\partial w_1$ through $g = w_1 - \varpi$ only. Its importance is due to the fact that it is still true when the elements are variable, but it must be remembered that the term $\partial R/\partial t$ is still a derivative with respect to t only as the latter is present through the coordinates of the disturbing planet.

(*d*) Since the term m'/a' in the preliminary expansion of R given in 4·2 does not contain the coordinates of the disturbed planet, R has the factor $m'a^2/a'^3$ when $a' > a$. This factor may be written $(m'/a)(a/a')^3$. Since the undisturbed force-function is μ/r, it follows that the disturbing effect of an outer planet has as factors the ratio of its mass to that of the sun and the cube of the ratio of the distances of the two planets from the sun.

If the outer planet be the disturbed planet, the first term of R can be included in the elliptic force-function. We can then consider R as having the square of the ratio of the distances as a factor instead of the cube.

6·6. *Elimination of a portion R_t of the disturbing function.*

In Chap. IV, R has been expanded into a sum of terms having the form

$$R = \Sigma K \cos N, \quad N = j_1 w_1 + j_2 w_2 + j_3 w_3 + j_1' w_1' + j_2' w_2', \quad \ldots(1)$$

where $\qquad w_1' = n't + \epsilon' = g' + \varpi', \quad w_2' = \varpi',$

and the j_i, j_i' are positive or negative integers or zero, the sign of summation referring to their various values. The coefficients K are functions of a, e, I, a', e' and therefore of c_1, c_2, c_3, a', e', and they contain the mass of the disturbing planet as a factor. Cf. 4·15.

Put
$$R = R_t + R_c, \quad \text{.......................(2)}$$
where R_t may contain any or all of the terms for which the relations $j_1 = j_1' = 0$ do not hold. Thus for all terms in R_t, the value of ν defined by
$$\nu = j_1 n + j_1' n', \qquad n = \mu^2/c_1^3, \quad \text{..............(3)}$$
is never zero. It is here assumed that n/n' is not in the ratio of two integers; this case, if it does occur, must also be excluded from R_t—it will be considered in Chaps. VIII, IX.

The elimination of R_t will be effected by a change of variables with the help of the Jacobian transformation theorem (5·3). The new variables will be denoted by
$$c_{i0},\ w_{i0}, \qquad i = 1, 2, 3,$$
and the suffix zero will in all cases denote that the old variables have been replaced by the new. Thus
$$N_0 = j_1 w_{10} + j_2 w_{20} + j_3 w_{30} + j_1' w_1' + j_2' w_2'. \quad \text{......(4)}$$

Take as the transforming function
$$\bar{S} = \Sigma_i c_i w_{i0} - S, \quad S = \Sigma \frac{K}{\nu} \sin N_0, \quad \text{.........(5)}$$

so that $\bar{S}$ is a function of the three old variables c_i and the three new variables w_{i0}; S is to contain only those terms present in R_t with the divisor ν appropriate to each term. It will be noticed that S can be regarded as the integral of R_t taken on the supposition that the ellipses are invariable; this fact, however, is not at this stage to be regarded as having any physical significance since (5) is merely a definition of S.

The equations 5·2 (3) give
$$c_{i0} = c_i - \frac{\partial S}{\partial w_{i0}}, \quad w_i = w_{i0} - \frac{\partial S}{\partial c_i}, \quad \text{...........(6)}$$

as the six relations connecting the old and new variables. Also

$$\frac{\partial \bar{S}}{\partial t} = \frac{\partial S}{\partial t} = -\Sigma \frac{j_1' n'}{\nu} K \cos N_0 = \Sigma \left(-1 + \frac{j_1 n}{\nu}\right) K \cos N_0, \quad \ldots(7)$$

by the definition (3) of ν. But when (5) is substituted in (6), the equation for c_{10} becomes

$$c_1 - c_{10} = \Sigma \frac{j_1}{\nu} K \cos N_0, \quad \ldots\ldots\ldots\ldots\ldots\ldots(8)$$

whence by (3)

$$\frac{\partial \bar{S}}{\partial t} = -\Sigma K \cos N_0 + \frac{\mu^2}{c_1^3}(c_1 - c_{10}).$$

The new Hamiltonian function is

$$R + \frac{\mu^2}{2c_1^2} + \frac{\partial \bar{S}}{\partial t} = R_c + \Sigma K \cos N - \Sigma K \cos N_0 + \frac{\mu^2}{2c_1^2} + \frac{\mu^2}{c_1^3}(c_1 - c_{10}). \quad \ldots\ldots(9)$$

The equations (5), (6) show that $w_i - w_{i0}$ has m' as a factor. Hence $\cos N - \cos N_0$ has the same factor, and $K \cos N - K \cos N_0$ has m'^2 as a factor. Also, since $c_1 - c_{10}$ has m' as a factor,

$$c_1^{-2} = (c_{10} + c_1 - c_{10})^{-2} = \frac{1}{c_{10}^2}\left\{1 - 2\frac{c_1 - c_{10}}{c_{10}} + 3\frac{(c_1 - c_{10})^2}{c_{10}^2} - \ldots\right\},$$

$$c_1^{-3} = \frac{1}{c_{10}^3}\left(1 - 3\frac{c_1 - c_{10}}{c_{10}} + \ldots\right).$$

The last two terms of (9) can therefore be written

$$\frac{\mu^2}{2c_{10}^2} - \frac{3(c_1 - c_{10})^2}{2c_{10}^4}\mu^2 + \ldots, \quad \ldots\ldots\ldots\ldots\ldots(10)$$

the first power of $c_1 - c_{10}$ disappearing. Finally, since $R_c - R_{c0}$ has the factor m'^2, the right-hand member of (9) may be written

$$R_{c0} + \frac{\mu^2}{2c_{10}^2} + \text{terms with factor } m'^2. \quad \ldots\ldots\ldots(11)$$

If, therefore, we omit terms having the factor m'^2, the equations satisfied by the new variables are

$$\Sigma (dc_{i0}\delta w_{i0} - dw_{i0}\delta c_{i0}) = dt \,.\, \delta\left(R_{c0} + \frac{\mu^2}{2c_{10}^2}\right). \quad \ldots(12)$$

These have the same form as the original equations, but the terms present in R_t have disappeared.

At this point it is convenient to indicate the general plan of the remaining portions of this chapter. Two choices of the terms to be included in R_t will be made. In one of them, R_t contains *all* the terms for which $\nu \neq 0$; R_c then contains only those terms which produce the so-called secular motions of the elements. In the other, R_t contains the short period terms only, so that R_c contains the long period and secular terms.

It is for the latter choice that the calculation of the terms in (10) dependent on m'^2 will be made. Several methods for carrying out the work to this degree of accuracy will be outlined, and certain cases where it is possible to obtain numerical results with but little calculation will be developed.

It may be pointed out that the work as far as equation (9) is quite general in character, no approximation being involved. It is only after this point that we assume the possibility of development in powers of m' and proceed to find the first terms of the expansion.

The First Approximation

6·7. *First approximation by change of variables.*

Let R_t contain all the terms for which $\nu \neq 0$; then R_c is independent of w_1, w_1' and therefore of t explicitly.

The first approximation is obtained by neglecting all terms which have m'^2 as a factor. The variables c_{i0}, w_{i0} therefore satisfy 6·6 (12). Since R_{c0} is obtained from R_c merely by substituting the new variables for the old, it follows that R_{c0} is independent of w_{10}, t. The equation for c_{10} therefore gives

$$\frac{dc_{10}}{dt} = 0, \qquad c_{10} = \text{const.} = k_1,$$

and the equation for w_{10} is

$$\frac{dw_{10}}{dt} = \frac{\mu^2}{c_{10}{}^3} - \frac{\partial R_{c0}}{\partial c_{10}} = \frac{\mu^2}{k_1{}^3} - \frac{\partial R_{c0}}{\partial c_{10}}.$$

When m' is neglected, the remaining variables are constant and then

$$w_{10} = \frac{\mu^2}{k_1{}^3}t + \alpha_1, \quad w_{20} = \alpha_2, \quad w_{30} = \alpha_3; \quad c_{i0} = k_i, \; \ldots\ldots(1)$$

where α_i, k_i are arbitrary constants; these values may be substituted in all terms containing the factor m' and therefore in R_{c0}. As R_{c0} does not contain w_{10}, it follows that all the derivatives

of R_{c0} will become constants. The solution of the equations 6·6 (12) to the order m' will therefore be

$$\left.\begin{aligned} c_{10} &= k_1, \quad w_{10} = \frac{\mu^2}{k_1^3} t - \frac{\partial R_{c0}}{\partial c_{10}} t + \alpha_1, \\ c_{i0} &= k_i + \frac{\partial R_{c0}}{\partial w_{i0}} t, \quad w_{i0} = \alpha_i - \frac{\partial R_{c0}}{\partial c_{i0}} t, \qquad i = 2, 3, \end{aligned}\right\} \quad \text{...(2)}$$

in which the coefficients of t are all constant.

The values of the new variables in terms of t having been obtained, the old variables are to be calculated from the equations 6·6 (6) with the value 6·6 (5) for S. But since S, $c_i - c_{i0}$ have the factor m' we can replace c_i by c_{i0} in S and in its derivatives. We therefore obtain

$$c_i = c_{i0} + \frac{\partial S_0}{\partial w_{i0}}, \quad w_i = w_{i0} - \frac{\partial S_0}{\partial c_{i0}}. \qquad \text{............(3)}$$

Finally, since the values (1) differ from (2) by terms having the factor m', we can replace w_{i0}, c_{i0} by their values (1) in S_0 and in its derivatives.

These results may be restated in the following manner. If

$$R = R_c + \Sigma K \cos N, \quad N = j_1 w_1 + j_2 w_2 + j_3 w_3 + j_1' w_1' + j_2' w_2',$$

where R_c contains all terms for which $j_1 = j_1' = 0$, the values of the variables c_i, w_i to the order m' are given by

$$\left.\begin{aligned} c_1 &= k_1 + \Sigma j_1 \frac{K}{\nu} \cos N, \\ w_1 &= \alpha_1 + \left(\frac{\mu^2}{k_1^3} - \frac{\partial R_c}{\partial c_1}\right) t - \Sigma \frac{\partial}{\partial c_1}\left(\frac{K}{\nu}\right) \sin N, \\ c_i &= k_i + \frac{\partial R_c}{\partial w_i} t + \Sigma j_i \frac{K}{\nu} \cos N, \\ w_i &= \alpha_i - \frac{\partial R_c}{\partial c_i} t - \Sigma \frac{1}{\nu} \frac{\partial K}{\partial c_i} \sin N, \end{aligned}\right\} \quad \text{...(4)}$$

where $i = 2, 3$; in all the terms of the right-hand members the constants k_1, k_2, k_3 are substituted for c_1, c_2, c_3, the constants α_2, α_3 for w_2, w_3, and the value $\alpha_1 + \mu^2 t/k_1^3$ for w_1, and ν is defined by

$$\nu = j_1 \frac{\mu^2}{k_1^3} + j_1' n'. \qquad \text{.....................(5)}$$

Equations (4) may be written in the following form. If we put

$$\psi = tR_c + \Sigma \frac{K}{\nu} \sin N, \quad \text{.................(6)}$$

$$c_i - k_i = \delta c_i, \quad w_1 - \alpha_1 - \frac{\mu^2}{k_1^3} t = \delta w_1, \quad w_i - \alpha_i = \delta w_i, \quad \text{...(7)}$$

then
$$\delta c_i = \frac{\partial \psi}{\partial w_i}, \quad \delta w_i = -\frac{\partial \psi}{\partial c_i}, \quad \text{.......................(8)}$$

the undisturbed values (1) being substituted in the right-hand members. These are called the perturbations of the elements. It must be remembered, however, that the coefficient of t in δw_1 is to be included with μ^2/k_1^3 in comparing with observation.

6·8. *Secular and periodic terms.*

The coefficient of t in w_1 is deduced from observation and is known as the 'observed mean motion.' If we denote it by n_{00}, we have

$$n_{00} = \frac{\mu^2}{k_1^3} - \left(\frac{\partial R_c}{\partial c_1}\right)_{c_1 = k_1}. \quad \text{..................(1)}$$

If we define a_{00} by means of the equation $a_{00}^3 n_{00}^2 = \mu$, we obtain, to order m',

$$\frac{k_1^2}{\mu} = a_{00}\left(1 - \frac{2}{3}\frac{1}{n_{00}}\frac{\partial R_c}{\partial c_1}\right)_{c_1 = k_1}. \quad \text{...........(2)}$$

But since $c_1^2 = \mu a$, the equation for c_1 gives, to the order m',

$$\frac{c_1^2}{\mu} = \frac{k_1^2}{\mu}\left(1 + \frac{2}{k_1} \Sigma j_1 \frac{K}{\nu} \cos N\right),$$

or
$$a = a_{00}\left(1 - \frac{2}{3}\frac{1}{n_{00}}\frac{\partial R_c}{\partial c_1} + \frac{2}{k_1}\Sigma j_1 \frac{K}{\nu}\cos N\right)_{c_1 = k_1}. \quad \text{...(3)}$$

Thus the mean value of a is not a_{00} but this quantity with a small additive portion.

The coefficients of t in c_2, c_3, w_2, w_3 are known as the secular parts of these variables. An important result is the fact that c_1, and therefore a, contains no secular part to the order m'.

Since the coefficient of t in w_1 is the observed mean motion, the secular part of w_1 is defined as any part which it may have depending on t^2, t^3, ... To the order m' there is no such part.

The coefficients of the periodic terms in w_1, w_2, w_3 depend on the derivatives with respect to c_1, c_2, c_3 of K/ν. Since ν is independent of c_2, c_3 but does depend on c_1, we have

$$\frac{\partial}{\partial c_i}\left(\frac{K}{\nu}\right)=\frac{1}{\nu}\frac{\partial K}{\partial c_i}, \qquad i=2,3,$$

$$\frac{\partial}{\partial c_1}\left(\frac{K}{\nu}\right)=\frac{1}{\nu}\frac{\partial K}{\partial c_1}-\frac{j_1 K}{\nu^2}\frac{\partial n}{\partial c_1}=\frac{1}{\nu}\frac{\partial K}{\partial c_1}+\frac{3j_1 nK}{\nu^2 c_1}, \qquad \text{......(4)}$$

since $n=\mu^2/c_1^3$.

The presence of the square of ν as a divisor in w_1 but in none of the other elements, is of fundamental importance in the theory of the long period terms which have small values of ν/n. The simple manner in which this divisor arises with the method of solution adopted here is noticeable.

6·9. *Transformation to the elements a, e_1, Γ_1 or a, e, Γ.*

In the developments of the disturbing function, the angles w_i, or quite simple linear functions of them, are used, but in place of the c_i we find the elements a, e_1, Γ_1, related to them by the formulae 5·15 (2), or the elements a, e, Γ, shown by 5·13 (1).

If f be any function of the c_i, we have, with arbitrary variations of them,

$$\Sigma\frac{\partial f}{\partial c_i}\delta c_i\equiv\delta f\equiv\frac{\partial f}{\partial a}\delta a+\frac{\partial f}{\partial e_1}\delta e_1+\frac{\partial f}{\partial \Gamma_1}\delta\Gamma_1, \qquad i=1,2,3. \quad \text{......(1)}$$

From 5·15 (2) we deduce

$$\delta a=\frac{2}{na}\delta c_1, \quad \delta e_1=-\frac{1}{na^2}\left(\frac{\delta c_2}{e_1}+\tfrac{1}{2}e_1\delta c_1\right), \quad \delta\Gamma_1=-\frac{1}{na^2}(\delta c_3+\Gamma_1\delta c_2). \quad \text{.........(2)}$$

On substituting these in (1) and equating the coefficients of the δc_i, we obtain

$$\left.\begin{aligned}\frac{\partial f}{\partial c_1}&=\frac{1}{na^2}\left(2a\frac{\partial f}{\partial a}-\tfrac{1}{2}e_1\frac{\partial f}{\partial e_1}-\Gamma_1\frac{\partial f}{\partial \Gamma_1}\right),\\ \frac{\partial f}{\partial c_2}&=-\frac{1}{na^2}\frac{1}{e_1}\frac{\partial f}{\partial e_1}, \quad \frac{\partial f}{\partial c_3}=-\frac{1}{na^2}\frac{\partial f}{\partial \Gamma_1}.\end{aligned}\right\} \quad \text{..................(3)}$$

The derivatives of K with respect to the c_i may be found from (3), and the differences of a, e_1, Γ_1 from constants by means of (2), if we put therein $\delta c_i=c_i-k_i$.

Similarly, for the elements a, e, Γ, we find

$$\left.\begin{aligned}\delta a&=\frac{2}{na}\delta c_1, \quad \delta e=-\frac{(1-e^2)^{\frac{1}{2}}}{na^2}\left\{\frac{1}{e}\delta c_2+\frac{e}{1+(1-e^2)^{\frac{1}{2}}}\delta c_1\right\},\\ \delta\Gamma&=-\frac{1}{na^2(1-e^2)^{\frac{1}{2}}}(\delta c_3+\Gamma\delta c_2+\Gamma\delta c_1),\end{aligned}\right\} \quad \text{......(4)}$$

and

$$\left.\begin{aligned}\frac{\partial f}{\partial c_1}&=\frac{1}{na^2}\left\{2a\frac{\partial f}{\partial a}-\frac{e(1-e^2)^{\frac{1}{2}}}{1+(1-e^2)^{\frac{1}{2}}}\frac{\partial f}{\partial e}-\frac{\Gamma}{(1-e^2)^{\frac{1}{2}}}\frac{\partial f}{\partial \Gamma}\right\},\\ \frac{\partial f}{\partial c_2}&=-\frac{1}{na^2}\left\{\frac{(1-e^2)^{\frac{1}{2}}}{e}\frac{\partial f}{\partial e}+\frac{\Gamma}{(1-e^2)^{\frac{1}{2}}}\frac{\partial f}{\partial \Gamma}\right\},\\ \frac{\partial f}{\partial c_3}&=-\frac{1}{na^2}\frac{1}{(1-e^2)^{\frac{1}{2}}}\frac{\partial f}{\partial \Gamma}.\end{aligned}\right\}\quad\text{.........(5)}$$

A difficulty occurs in consequence of the presence of the divisor e_1 or e in the expressions for $\partial f/\partial c_2$. But this latter derivative is present only in the expression 6·7 (4) for w_2, which in turn only appears in terms having the factor e in the expressions for the longitude, latitude, and radius vector in terms of the elliptic elements. It is for this reason that it is usual to give the perturbation of $\delta\varpi$ in the form $e\delta\varpi$.

This solution of the difficulty is sufficient when we confine ourselves to the first power of m', but further consideration is necessary when we proceed to higher powers. The solution is then contained in the theorem of 6·4, applied to the development given in 6·15.

6·10. *The perturbations of the coordinates.*

The coordinates are supposed to be expressed in terms of the elliptic elements. If, then, we define the perturbations of the elements as in 6·7, the perturbations of any coordinate x to the first power of m' will be given by

$$\delta x=\Sigma\left(\frac{\partial x}{\partial c_i}\delta c_i+\frac{\partial x}{\partial w_i}\delta w_i\right)=\Sigma\left(\frac{\partial x}{\partial c_i}\frac{\partial \psi}{\partial w_i}-\frac{\partial x}{\partial w_i}\frac{\partial \psi}{\partial c_i}\right),\qquad i=1,\,2,\,3.$$

The latter form again introduces a function of the type considered in 6·4. The periodic part S of ψ is a d'Alembert series and the longitude, latitude, and radius vector, when expressed in terms of the elliptic elements are d'Alembert series. It follows that the periodic parts of the perturbations of these three coordinates are also d'Alembert series. It should be remembered, however, that $\delta c_2/e$, $e\delta w_2$ contain perturbations which do not vanish with e.

6·11. This form of the solution, in which powers as well as periodic functions of t are present, is that which is usual in applications to problems of the solar system. The results in this form usually have sufficient accuracy during the limited intervals over which observations are available. It is evident that a continuation of the process to powers of m' beyond the first

will lead to terms containing higher powers of t and to the presence of periodic functions of t multiplied by powers of t. That it is possible formally to express the perturbations wholly by periodic functions of t, at least in a first approximation, may be indicated in the following manner.

Instead of the variables c_2, w_2, let us use p_2, q_2 defined in 5·14, and for simplicity let us neglect the inclination. The canonical equations for these two variables are

$$\frac{dp_2}{dt}=\frac{\partial R}{\partial q_2}, \quad \frac{dq_2}{dt}=-\frac{\partial R}{\partial p_2}. \qquad \text{(1)}$$

The process for eliminating the terms present in R_t can still be followed: it leads to new variables p_{20}, q_{20} with the same function R_{c0}, which is the portion of R independent of w_1, w_1'. The development of R as far as the second powers of the eccentricities is given by 4·32 (2), and the 'non-periodic part,' depending on the eccentricities, is given by putting $i=0$ in the first line of the expression, and $i=-1$ in the second line. The resulting terms have the form*

$$\{\tfrac{1}{2}P(e^2+e'^2)+Qee'\cos(\varpi-\varpi')\}\frac{m'}{a'}, \qquad \text{(2)}$$

where P, Q are functions of a/a'. Just as before we show that c_1 and therefore a is constant. To the second powers of the eccentricities, we have, by 5·14,

$$e\sin\varpi=p_2/c_1, \quad e\cos\varpi=q_2/c_1.$$

Since a is constant we can use units such that μ, a, n, are all unity. If we put $e'\sin\varpi'=p'$, $e'\cos\varpi'=q'$, the expression (2) may be written

$$\{\tfrac{1}{2}P(p_2^2+q_2^2+p'^2+q'^2)+Q(pp'+qq')\}m'\alpha.$$

With this portion of R the canonical equations (1) give

$$\frac{dp_2}{dt}=(Pq_2+Qq')m'\alpha, \quad \frac{dq_2}{dt}=-(Pp_2+Qp')m'\alpha,$$

leading to

$$\frac{d^2p_2}{dt^2}=-(P^2p_2+PQp')m'^2\alpha^2, \quad \frac{d^2q_2}{dt^2}=-(P^2q_2+PQq')m'^2\alpha^2,$$

and furnishing the solution,

$$p_2=-\frac{Q}{P}p'+C\sin(m'\alpha Pt+D), \quad q_2=-\frac{Q}{P}q'+C\cos(m'\alpha Pt+D),$$

where C, D are arbitrary constants.

With the adopted units, the period of revolution of the planet is 2π. Since $\alpha P<1$ and $m'<\cdot001$ for the largest planet Jupiter, it follows that the period of the periodic parts of p_2, q_2 is very long compared with the period of revolution of the planet—in general, it is greater than 10,000 years. Hence, for intervals of a few centuries, we can develop these periodic

* As α contains e^2, e'^2, functions of α must be expanded as far as e^2, e'^2 in powers of these parameters.

terms in powers of t and still obtain the needed accuracy without additional calculation.

The practical objection to this method of procedure is the complication caused by the introduction of a new argument. Still another argument would be introduced by the solution of the equations for p_3, q_3. Thus, four arguments would be present. Even with a single disturbing planet and the calculation confined to the first power of m', the work becomes complicated as soon as we proceed beyond the second powers of the eccentricities and inclination and the labour becomes almost prohibitive in the case, for example, of the mutual perturbations of Jupiter and Saturn.

In satellite theories, the periods are so short that expansions in powers of t are impracticable and the four arguments must be retained. On the other hand only a few powers of α, and consequently a few values of i, in the expansions of Chap. IV, are needed. Part of the compensation thus afforded is lost by the fact that many powers of the disturbing mass must be retained.

The general theory of the secular motions of the elements, to which the solution just given constitutes an introduction, will not be given in this volume. The reader is referred to other treatises, particularly to that of Tisserand, vol. 1, Chap. XXVI, and to later work referred to in *Ency. Math. Wiss.* Bd. 5. A warning concerning frequently quoted results, which give limits to the eccentricities and inclinations, should be made. These investigations, in general, take into account the first powers of the disturbing masses and the earlier powers of the eccentricities and inclinations only. When higher powers are included, approximate resonance conditions have to be considered and these may alter the limits extensively over very long periods of time. Thus while such results may be used with a fair degree of confidence in cosmogonic speculations for a few million years from the present time, we have no present knowledge as to the facts over intervals of the order of 10^9 years.

6·12. *Long period terms.*

We have seen that in a first approximation we may substitute elliptic values for the coordinates in the development of the disturbing function. In Chap. IV, this development gives any angle N in the form

$$i(w_1 - w_1') \pm jg \pm j'g' \pm k(w_1 + w_1' - 2\theta),$$

where i, j, j', k are positive integers or zeros. The coefficient of a term with this argument contains the factor (cf. 4·15)

$$e^j e'^{j'} \Gamma^k.$$

When elliptic values are used, the coefficient of t in the argument is

$$\nu = i(n - n') \pm jn \pm j'n' \pm k(n + n').$$

It follows that if we have any argument in which the coefficient of t is $j_1 n \pm j_1' n'$, the order of its coefficient will be $|j_1 \pm j_1'|$, as far as powers of the eccentricities and inclination are concerned.

A long period term is defined as one in which

$$\nu/n = (j_1 n \pm j_1' n') \div n$$

is small compared with unity: the secular terms for which $j_1 = j_1' = 0$ are excluded from this definition. Since it is supposed that we are using the methods developed above, n should properly be replaced by n_{00}, but the suffix may be omitted in the discussion. Since n, n' are usually positive we have to consider cases where n'/n is nearly equal to the ratio of two integers. The word 'small' as used above is indefinite in both the theory and its applications; in general, if ν/n is less than about one-third, so that the period of the term is longer than three times that of the revolution of the body round the sun, the term would be treated as one of long period.

Since n, n' are obtained independently from observation it is always possible to find terms in R for which ν/n is small. The critical values of j_1, j_1' are obtained by expanding n'/n as a continued fraction. If p/q be any convergent, then for any convergent after the first, $j_1 = p$, $j_1' = q$ will give a long period term. But as the order of the coefficient is $|p - q|$ such a term may be quite insensible to observation even after receiving the factor $1/\nu^2$.

For example in the case of Jupiter and Saturn where $n = 43996''$ (Saturn), $n' = 109256''$ (Jupiter), these being the mean annual motions, we have

$$2n - n' = -\cdot 483n, \quad \text{order } 1;$$

$$5n - 2n' = \cdot 0334n, \quad \text{order } 3;$$

$$72n - 29n' = -\cdot 0162n, \quad \text{order } 43.$$

The third of these is obviously insensible.

In considering the degree of approximation needed, we can make use of the property of a continued fraction which states that if p/q, p'/q' are consecutive convergents, $p' > p$, no fraction whose denominator lies between q, q' gives so close an approximation as p/q. Thus if $q' - q$ is large, p/q is usually a close approximation, ν/n is very small and higher convergents are unlikely to give sensible coefficients. An apparent exception to the argument is the case of multiples of N when the term with argument N has a sensible coefficient. This case is dealt with in the second approximation (cf. 6·18). In the case of Jupiter and Saturn, the term for which $\nu = 10n - 4n'$ has a sensible coefficient; that for which $\nu = 15n - 6n'$ is insensible.

We have seen that the element w_1 is that chiefly affected by a long period term since the term in this element has the divisor ν^2, while those in the other elements have the divisor ν only. In other words, it is the mean longitude which shows the principal effect. But there is an associated short period term in the true longitude which may have a coefficient comparable in magnitude with that of the long period term and which arises in the following way. The determination of the perturbations of e, ϖ, or more properly, of $e \cos \varpi$, $e \sin \varpi$ substantially requires the division of the term in the disturbing function by e (6·9). Thus the term in R substantially acquires the divisor $e\nu$ when it is inserted in the true longitude. Now e, ϖ occur in the true longitude principally through the chief elliptic term $2e \sin (w_1 - \varpi)$ and the long period term, therefore, gives two terms with motions $n \pm \nu$. As ν is small, these have nearly the period of revolution as periods. One of the two coefficients is usually quite small owing to the association of powers of e with multiples of ϖ: the proof of this statement is simple and is left to the reader.

The fact that terms of very long period are usually of very high order would seem to suggest some theoretical limit beyond which such terms could always be neglected. But, as far as is known, the observational determinations of the constants which give the mean motions are substantially independent of those which give the eccentricities, when the

determination is spread over many revolutions of the planet round the sun. Thus the ratio of $e^j e'^{j'} \Gamma^k$ to ν or ν^2 has no definite limit and a coefficient may be very small or very large according to the values chosen for n, n'. From the practical point of view the difficulty is surmounted by supposing that such terms, having periods very long compared with the interval of observation, can be expanded in powers of t. The constant parts of the expansions are absorbed in the arbitrary constants of the solution and, in the case of the true longitude, the coefficient of t is absorbed in the mean motion; the remaining portions are usually quite insensible during the interval.

There is, however, an upper limit because of the fact that when the coefficient of the term exceeds a certain magnitude comparable with 2π, the procedure previously followed becomes invalid: the phenomena of resonance then begin to appear. This limitation does not remove the difficulty. The form of the mathematical development has to be changed and the argument proceeds on different lines. The complications which arise make the problem exceedingly difficult: some indications of them will be given in Chap. VIII. It may be pointed out that one of the difficulties is due to the fact that the period of the term may become comparable with the periods of the so-called secular terms (6·8) and that it is not then possible to treat them independently even in a first approximation.

6·13. *Other forms of solution.*

Instead of the value of $\bar{S}$ written in 6·6, we might have used

$$\bar{S} = \Sigma c_{i0} w_i + \Sigma \frac{K_0}{\nu_0} \sin N.$$

Here the rôles of c_i, w_i are simply interchanged. A little consideration will make evident the fact that in a first approximation this form of $\bar{S}$ gives nothing new: it is only in the second and higher approximations that differences appear.

When the Poincaré variables p_2, q_2, p_3, q_3 are used, the expansion of R takes the form

$$R = \Sigma K_c \cos N' + \Sigma K_s \sin N',$$

where $$N' = j_1 w_1 + j_1' w_1' + \beta,$$

β depending on the constants of the disturbing body only. In this form K_c, K_s are expanded in positive integral powers of

p_2, q_2, p_3, q_3 and are also functions of c_1, a', e'. The transformation function to be used is then

$$S_1 = c_1 w_{10} + p_2 q_{20} + p_3 q_{30} - \Sigma \frac{(K_c)_0}{\nu} \sin N_0' + \Sigma \frac{(K_s)_0}{\nu} \cos N_0',$$

where the suffix zero has the meaning defined in 6·6 except in $(K_c)_0$, $(K_s)_0$, where it means that q_2, q_3 are replaced by q_{20}, q_{30}, while p_2, p_3 are left unchanged. We may also interchange the rôles played by p, q.

THE SECOND APPROXIMATION

6·14. The calculation of the second approximation to the values of the variables may be very laborious if advantage is not taken of every feature which may help to shorten it. The first step consists of an examination to discover what classes of terms will give sensible coefficients or sensible additions to coefficients found in the first approximation. Methods for the calculation of the sensible portions will then be given and these methods will be developed in such a manner that the actual computation may be reduced to comparatively few operations.

There are several devices which can be used to obtain the terms dependent on the squares of the disturbing masses. We can follow the process of Delaunay which involves continual changes of variables until the Hamiltonian function is freed from all sensible terms for which the relations $j_1 = j_1' = 0$ do not hold; the equations for the final variables can then be solved by series arranged along powers of t. Another plan is the substitution of the results of the first approximation in the derivatives of R instead of the elliptic values which have been used in finding the first approximation; the equations are then again integrated and the new portions of the variables calculated. Another device which is sometimes useful is a method of integration by parts which makes use of the fact that the derivatives of c_1, c_2, c_3, w_2, w_3 with respect to t have m' as a factor.

The most useful device is, however, the separation of the terms of long period from those of short period and also from

those which are secular. It will be seen that the second approximations to the long period and to the secular terms can be rendered almost independent of the first approximation to the short period terms, so that they can be determined independently of the latter. But the effects of the long period and secular terms on the short period terms are usually sensible, and it is these effects which become most evident in comparisons with observations extending over long intervals of time.

The methods adopted to prove that these limitations are possible are not necessarily the most convenient for the actual calculation of the sensible terms, so that more than one of the plans for continuing the approximations will be found developed in the sections which follow.

6·15. *The Hamiltonian function in terms of* c_{10}, w_{10}.

The equations 6·6 (9), 6·6 (10) give, for the Hamiltonian function of the equations for the new variables, the value

$$\frac{\mu^2}{2c_{10}^2} - \frac{3}{2}\frac{\mu^2}{c_{10}^4}(c_1 - c_{10})^2 + R_c + \Sigma K \cos N - \Sigma K \cos N_0, \qquad \text{......(1)}$$

in which powers of $c_1 - c_{10}$ beyond the second are dropped. This last omission is equivalent to stopping at the order m'^2. We need (1) expressed in terms of c_{i0}, w_{i0} to this order.

Since S has the factor m', we have $c_i = c_{i0}$, $w_i = w_{i0}$ when $m' = 0$. It follows that, as far as the first power of m', the equations 6·6 (6) give

$$c_i = c_{i0} + \frac{\partial S_0}{\partial w_{i0}}, \quad w_i = w_{i0} - \frac{\partial S_0}{\partial c_{i0}}, \qquad \text{...........(2)}$$

and that, if f be any function of c_i, w_i, t,

$$f(c_i;\ w_i;\ t) = f(c_{i0};\ w_{i0};\ t) + \Sigma \frac{\partial f}{\partial c_i}(c_i - c_{i0}) + \frac{\partial f}{\partial w_i}(w_i - w_{i0})$$

to the same order. The substitution of (2) in this last equation enables us to write it

$$f = f_0 + \Sigma\left(\frac{\partial f_0}{\partial c_{i0}}\frac{\partial S_0}{\partial w_{i0}} - \frac{\partial f_0}{\partial w_{i0}}\frac{\partial S_0}{\partial c_{i0}}\right). \qquad \text{...........(3)}$$

On applying this to $\Sigma K \cos N = R_t$ and to $\Sigma K \cos N_0$, we obtain

$$\Sigma K \cos N = R_{t0} + \Sigma \left(\frac{\partial R_{t0}}{\partial c_{i0}} \frac{\partial S_0}{\partial w_{i0}} - \frac{\partial R_{t0}}{\partial w_{i0}} \frac{\partial S_0}{\partial c_{i0}} \right), \quad \ldots\ldots(4)$$

$$\Sigma K \cos N_0 = R_{t0} + \Sigma \left(\frac{\partial R_{t0}}{\partial c_{t0}} \frac{\partial S_0}{\partial w_{i0}} \right). \quad \ldots\ldots\ldots\ldots\ldots\ldots(5)$$

Since R already contains the factor m', these results hold to the order m'^2. The same application to the development of R_c may be made. The second term of (1) evidently has the factor m'^2 and may be replaced by its value in terms of the new variables by means of the first of equations (2).

Thus the Hamiltonian function for the canonical equations satisfied by c_{i0}, w_{i0} to the order m'^2 becomes

$$\frac{\mu^2}{2c_{10}^2} + R_{c0} + F_1 + F_t + F_c, \quad \ldots\ldots\ldots\ldots\ldots\ldots(6)$$

where

$$\left.\begin{aligned} F_1 &= -\frac{3}{2}\frac{\mu^2}{c_{10}^4}\left(\frac{\partial S_0}{\partial w_{10}}\right)^2, \quad F_t = -\Sigma \left(\frac{\partial R_{t0}}{\partial w_{i0}} \frac{\partial S_0}{\partial c_{i0}} \right), \\ F_c &= \Sigma \left(\frac{\partial R_{c0}}{\partial c_{i0}} \frac{\partial S_0}{\partial w_{i0}} - \frac{\partial R_{c0}}{\partial w_{i0}} \frac{\partial S_0}{\partial c_{i0}} \right). \end{aligned}\right\} \quad \ldots(7)$$

The expressions (7) contain the factor m'^2. The application of the theorem of 6·4 shows that they are d'Alembert series and therefore that they contain no powers of e, Γ as divisors.

If R_t be defined to contain *all* the terms in which $j_1 = j_1' = 0$ do not simultaneously hold and no others, R_c and therefore R_{c0} will contain only the terms in which these relations do hold. Since S_0 contains the same terms as those present in R_{t0}, it follows that F_c is like R_{t0} in this respect, and therefore that the secular portions which depend on the terms in which $j_1 = j_1' = 0$ will arise only from F_1, F_t, R_{c0}.

The Hamiltonian function (6) will, however, be used below only to distinguish between the effects of the short period and long period terms, and for this purpose R_t will be defined to contain the short period terms only. The investigation will show what portions may be neglected in the actual calculations which can then be carried out by a more simple method.

A direct second approximation to the solution of the equations for c_i, w_i, is easily seen to be given by

$$\left.\begin{aligned} c_i &= c_{i0} + \frac{\partial S_0}{\partial w_{i0}} + \Sigma_j \frac{\partial^2 S_0}{\partial w_{i0}\partial c_{j0}} \frac{\partial S_0}{\partial w_{j0}}, \\ w_i &= w_{i0} - \frac{\partial S_0}{\partial c_{i0}} - \Sigma_j \frac{\partial^2 S_0}{\partial c_{i0}\partial c_{j0}} \frac{\partial S_0}{\partial w_{j0}}. \end{aligned}\right\} \quad \text{............(8)}$$

These require the formation of the products of the derivatives of S_0 for each of the six variables. The method given in the text confines the formation of such products to those in one function, namely (6).

6·16. *Influence of the short period terms in the first approximation on the second approximation.*

Let R_t and therefore S_0 contain only short period terms, so that in 6·15 (6) there are no small divisors ν or ν^2 tending to raise the magnitudes of the terms in F_1, F_t, F_c. Suppose that all these functions are expressed as sums of periodic terms. These terms will have the same general form with respect to the variables c_{i0}, w_{i0} that R had with respect to c_i, w_i, that is, they have the form $K_0 \cos N_0$, where N_0 is a linear function of the w_{i0}, w_1', w_2' with integral coefficients, and K_0 is a function of the c_{i0} and of the elements of the disturbing planet.

The terms present in R_{c0} are all either of long period or those for which $j_1 = j_1' = 0$, while those arising from F_1, F_t, F_c are of the same character with additional terms of short period but all having the factor m'^2.

If the short period terms were again eliminated by a Jacobian transformation, the new variables would differ from c_{i0}, w_{i0} by terms having the factor m'^2 and with no small divisors present. As the largest value of m' in the problems of the solar system is less than ·001, and as an accuracy to ·001 of a short period coefficient is rarely attainable in comparisons with observations, these portions can generally be neglected.

The long period terms present in F_1, F_t—there are none in F_c because the product of terms of the form $\cos(at + a')$,

$\cos(At+A')$, in which a is small and A is of the order of the mean motion, gives rise to terms with arguments

$$(A \pm a)\,t + A' \pm a'—$$

have the factor m'^2 and are additive to those in R_{c0} with the factor m'. Thus these terms will merely change the coefficients of the long period terms by amounts of the order of ·001 of their values at most, and such changes again are rarely sensible to observation. The same result holds for the terms in which $j_1 = j_1' = 0$.

Exception to these statements may arise on account of the fact that when we differentiate with respect to c_{20} or c_{30} a divisor of order e^2 or Γ is, in fact, introduced. In the method adopted for calculation it is seen (last paragraph of 6·17) that these terms of lower order disappear, so that the general argument is not affected by them.

Thus the short period terms present in the first approximation can be altogether neglected in proceeding to a second approximation, or at most, only a very few, and those with the largest coefficients, need be retained. It follows that the long period and secular terms can be obtained to the order m'^2 with sufficient accuracy if we neglect at the outset nearly all short period terms present in R.

As the apparent exception mentioned in the text always raises a difficulty in the discussion of the canonical equations for the elements, further details as to the occurrence of such terms may be of value.

According to the theorem of 6·4, R_c produces d'Alembert series in F_c and is therefore free from these terms of lower order. Hence they will only arise through F_t. The divisor e^2 will arise in F_t only through the product $(\partial R_{t0}/\partial w_{20})\,(\partial S_{t0}/\partial c_{20})$. This may be written $P+Q$, where

$$P,\,Q = \mp \frac{1}{2}\frac{\partial R_{t0}}{\partial c_{20}}\frac{\partial S_0}{\partial w_{20}} + \frac{1}{2}\frac{\partial R_{t0}}{\partial w_{20}}\frac{\partial S_0}{\partial c_{20}}.$$

By the theorem just quoted, P is a d'Alembert series, and is therefore free from the exceptional terms. As for Q, we note that elliptic values are to be substituted and that then $R_{t0} = dS_0/dt$. Hence, since derivatives with respect to t, w_{20}, c_{20} are commutable we deduce

$$Q = \frac{1}{2}\frac{d}{dt}\left(\frac{\partial S_0}{\partial c_{20}}\frac{\partial S_0}{\partial w_{20}}\right).$$

On account of the relation between R, S this suggests that some second order terms might have been included in the expression for S which would have prevented the occurrence of these terms in the new Hamiltonian functions. The fact that another method shows that they ultimately disappear, indicates that the portions of this character which arise from the solution of the equations 6·15 (8) which give c_i, w_i in terms of c_{i0}, w_{i0} to the second order, will cancel the portions which arise through Q.

As a matter of fact, even if the method were used for calculation, the terms would cause very little trouble. For we are actually interested only in the long period terms present in Q, and the operator d/dt introduces the small factor ν in such terms. The numerical effect of this fact would be to cancel to a large extent that produced by the divisor e^2 or Γ.

6·17. *Calculation of the second approximation to long period and secular terms.*

The fact that the first approximation to the short period terms exercises little or no sensible influence on the second approximation to the secular and long period terms, enables us to calculate the latter as though the former did not exist. Thus the equations for c_{i0}, w_{i0} become the same as the original equations for c_i, w_i would have been if we had omitted all short period terms. For the sake of brevity in notation, therefore, we shall omit the suffix zero in finding this second approximation, restoring it only at the end of the work.

The equations are

$$\frac{dc_i}{dt} = \frac{\partial}{\partial w_i}\left(\frac{\mu^2}{2c_1^2} + R\right), \quad \frac{dw_i}{dt} = -\frac{\partial}{\partial c_i}\left(\frac{\mu^2}{2c_1^2} + R\right), \quad \ldots(1)$$

all short period terms being excluded from R. These equations, except that for w_1, may be written

$$\frac{dc_i}{dt} = \frac{\partial R}{\partial w_i}, \quad \frac{dw_2}{dt} = -\frac{\partial R}{\partial c_2}, \quad \frac{dw_3}{dt} = -\frac{\partial R}{\partial c_3}. \quad \ldots\ldots(2)$$

If we differentiate the equation for w_1 and make use of the first of equations (2) we obtain

$$\frac{d^2 w_1}{dt^2} = \frac{d}{dt}\left(\frac{\mu^2}{c_1^3} - \frac{\partial R}{\partial c_1}\right) = -\frac{3\mu^2}{c_1^4}\frac{\partial R}{\partial w_1} - \frac{d}{dt}\left(\frac{\partial R}{\partial c_1}\right). \quad \ldots\ldots(3)$$

The first approximation gives the values of c_i, w_i in the form

$$\beta_0 + \beta_1 t + \Sigma B \cos(\nu t + \nu_0), \quad \ldots\ldots\ldots\ldots\ldots(4)$$

if we note that the addition of $\frac{1}{2}\pi$ to ν_0 will take care of the presence of sines. The coefficients β_1, B all contain the factor m' except that of t in w_1 which is n_{00}. In the case of c_1, we have $\beta_1 = 0$. Finally, B contains the first power of ν as a divisor in all cases except that of w_1 which contains an additional part having ν^2 as a divisor.

This first approximation was obtained on the supposition that the elliptic elements in R were constant. If we denote the difference of the constant and variable values of the elements by the symbol δ (in the case of w_1, the symbol δw_1 denotes the difference between w_1 and its undisturbed value $n_0 t + \text{const.}$), and if the additional part due to the second approximation be denoted by δ_2, Taylor's theorem, applied to equations (2), (3), gives

$$\frac{d}{dt}\delta_2 c_i = \Sigma_j \left(\frac{\partial^2 R}{\partial w_i \partial w_j}\delta w_j + \frac{\partial^2 R}{\partial w_i \partial c_j}\delta c_j\right), \quad i = 1, 2, 3, \quad \text{...(5)}$$

$$\frac{d}{dt}\delta_2 w_i = -\Sigma_j \left(\frac{\partial^2 R}{\partial c_i \partial w_j}\delta w_j + \frac{\partial^2 R}{\partial c_i \partial c_j}\delta c_j\right), \quad i = 2, 3, \quad \text{...(6)}$$

$$\frac{d^2}{dt^2}\delta_2 w_1 = \frac{12\mu^2}{c_1^5}\frac{\partial R}{\partial w_1}\delta c_1 - \frac{3\mu^2}{c_1^4}\Sigma_j \left(\frac{\partial^2 R}{\partial w_1 \partial w_j}\delta w_j + \frac{\partial^2 R}{\partial w_1 \partial c_j}\delta c_j\right)$$
$$- \frac{d}{dt}\Sigma_j \left(\frac{\partial^2 R}{\partial c_1 \partial w_j}\delta w_j + \frac{\partial^2 R}{\partial c_1 \partial c_j}\delta c_j\right), \quad \text{......(7)}$$

in which j takes the values 1, 2, 3. Since all the terms in the right-hand members of these equations have the factor m'^2, constant values may be substituted for the elements in the derivatives of R. Also since δc_i, δw_i are present in a linear form only, their various portions may be separately calculated in any manner which may be convenient.

According to 6·7 (8) the periodic parts of δc_i, δw_i are found from

$$\delta c_i = \frac{\partial S}{\partial w_i}, \quad \delta w_i = -\frac{\partial S}{\partial c_i}, \quad \text{..............(8)}$$

where $$S = \Sigma \frac{K}{\nu}\sin N, \quad R_t = \Sigma K \cos N, \quad \text{...........(9)}$$

with $$N = j_1 w_1 + j_2 w_2 + j_3 w_3 + j_1' w_1' + j_2' w_2',$$
$$\nu = j_1 n + j_1' n', \quad \nu \neq 0, \quad n = \mu^2/c_1^3, \quad \text{.........(10)}$$

constant values for the elements being used in these expressions.

Hence

$$\delta c_i = \Sigma \frac{j_i}{\nu} K \cos N, \quad \delta w_2 = -\Sigma \frac{1}{\nu}\frac{\partial K}{\partial c_2} \sin N, \quad \delta w_3 = -\Sigma \frac{1}{\nu}\frac{\partial K}{\partial c_3} \sin N, \quad \text{......(11)}$$

$$\delta w_1 = -3\Sigma \frac{K}{\nu^2}\frac{n}{c_1} \sin N - \Sigma \frac{1}{\nu}\frac{\partial K}{\partial c_1} \sin N. \quad \text{......(12)}$$

Although these formulae serve for all periodic terms, we are considering only the long period terms in R_t.

The secular parts are given by

$$\left.\begin{aligned} &\delta c_1 = 0, \quad \delta c_2 = t\frac{\partial R_c}{\partial w_2}, \quad \delta c_3 = t\frac{\partial R_c}{\partial w_3}, \\ &\delta w_1 = 0, \quad \delta w_2 = -t\frac{\partial R_c}{\partial c_2}, \quad \delta w_3 = -t\frac{\partial R_c}{\partial c_3}, \end{aligned}\right\} \quad \text{......(13)}$$

where $$R_c = \Sigma K \cos(j_2 w_2 + j_3 w_3 + j_2' w_2').$$

The substitution of (8) in the right-hand member of (5) gives

$$-\Sigma_j \frac{\partial^2 R}{\partial w_i \partial w_j}\frac{\partial S}{\partial c_j} + \Sigma_j \frac{\partial^2 R}{\partial w_i \partial c_j}\frac{\partial S}{\partial w_j}.$$

According to the theorem of 6·4, this is a d'Alembert series since S, R, $\partial R/\partial w_i$ are d'Alembert series. The same result is evidently true for the series in (7). For $i = 2$ in (6), we note that $c_2 \partial R/\partial c_2$ is a d'Alembert series, so that $c_2 \delta_2 w_2$ has the same character. Similarly $c_3 \delta_2 w_3$ is such a series also. Thus the presence of the divisor e^2 or Γ, lowering the orders of certain terms in the first approximation, does not affect the equations for the second approximation because such terms disappear. This is the proof referred to in 6·16.

6·18. *The principal part of the coefficient of a second order term.*

Let us consider first the case of a single term of long period. In general, the principal part of the coefficient will be that part

in which the divisor ν occurs to the highest power. This is evidently the fourth, obtained by substituting the first term of δw_1 in 6·17 (12) in the first line of 6·17 (7). It gives

$$\frac{d^2}{dt^2}(\delta_2 w_1) = \frac{3\mu^2}{c_1^4}\frac{\partial^2 R}{\partial w_1^2} \cdot \frac{3K}{\nu^2} j_1 \frac{n}{c_1} \sin N$$
$$= -9j_1^3 \frac{K^2}{\nu^2}\frac{n^2}{c_1^2} \sin N \cos N.$$

Whence, integrating,

$$\delta_2 w_1 = \frac{9}{8} j_1^3 \frac{K^2}{\nu^4}\frac{n^2}{c_1^2} \sin 2N.$$

If, then, we write

$$\delta w_1 = B \sin N,$$

for the first term in 6·17 (12), we obtain

$$\delta_2 w_1 = \frac{j_1}{8} B^2 \sin 2N. \qquad \text{.................(1)}$$

This result is independent of the method by which the coefficient B may have been obtained. It gives at once the principal part of the second order term with argument $2N$ when the first order part with argument N is known*.

It still holds if we include in δw_1 all the terms for which j_1, j_1' are the same. For these terms may be written in the form

$$P \cos(j_1 w_1 + j_1' w_1') + Q \sin(j_1 w_1 + j_1' w_1'),$$

where P, Q are functions of c_1, c_2, c_3, w_2, w_3, a', e', w_2', and this expression may be put into the form

$$B \sin(j_1 w_1 + j_1' w_1' + \beta),$$

where β, B are independent of w_1, w_1'. In this case we put $j_1 w_1 + j_1' w_1' + \beta$ for N. It is thus immediately applicable when the numerical values of B, β have been obtained.

It is not difficult to extend this result to the case in which two or more long period terms are present. For two such terms in R denoted by $K \cos N$, $\bar{K} \cos \bar{N}$, we have

$$\frac{\partial^2 R}{\partial w_1^2} = -j_1^2 K \cos N - \bar{j}_1^2 \bar{K} \cos \bar{N}.$$

* The result, obtained by a different method, was given by E. W. Brown, *loc. cit.* 7·32.

If
$$\delta w_1 = B\cos N + \bar{B}\cos\bar{N},$$
we have
$$B = -3Kn/\nu^2 c_1, \quad \bar{B} = -3\bar{K}n/\bar{\nu}^2 c_1.$$
Whence, on integration,
$$\delta_2 w_1 = \tfrac{1}{2}B\bar{B}\,\frac{\pm j_1\nu^2 + j_1\bar{\nu}^2}{(\pm\nu+\bar{\nu})^2}\sin(N+\bar{N}). \quad \ldots\ldots\ldots(2)$$

In the general case, we add together all such pairs of terms.

6·19. *Effect of a long period perturbation of the disturbing planet.*

The effect of such a perturbation is most marked in w_1' and the principal part of its effect on a term $K\cos N$ in R will be
$$\frac{\partial R}{\partial w_1'}\,\delta w_1' = -j_1'\,K\sin N\,.\,\delta w_1'.$$

In the formation of the canonical equations, it is assumed that w_1' is independent of the elements of the disturbed planet. Hence
$$\frac{\partial}{\partial w_1}\left(\frac{\partial R}{\partial w_1'}\,\delta w_1'\right) = -j_1 j_1'\,K\cos N\,.\,\delta w_1'.$$

If we are given
$$\delta w_1' = B'\sin N' = B'\sin(\nu' t + \nu_0'),$$
where ν'/n is small, we obtain, by a procedure similar to that followed in 6·18,
$$\delta_2 w_1 = \tfrac{1}{2}BB'\,\frac{\nu^2 j_1'}{(\pm\nu+\nu')^2}\sin(\pm N + N'). \quad \ldots\ldots(1)$$

When we are dealing with the mutual perturbations of two planets, there will be terms in $\delta w_1'$, due to the effect of the planet (which we have been calling the disturbed planet) having the argument N, and for these terms (1) gives
$$\delta_2 w_1 = \frac{j_1'}{2}\,BB'\sin 2N.$$

This case requires care if it is deduced directly rather than by substitution in (1), because $\delta w_1'$ contains the elements c_i, w_i, and we might be tempted to substitute in R before forming the

derivatives with respect to c_i, w_i. That we cannot do so is seen from the statement made above, namely, that the canonical equations are formed on the basis that the coordinates of the disturbing planet are functions of t only and are independent of the elements of the disturbed planet. This basis must be retained in the subsequent work.

A useful exercise for the student is the deduction of these results by solving the equations of 6·15. He will find, for example, that the solution 6·15 (8) of the equations 6·6 (6) to the second order will contribute $-j_1 B^2 \sin 2N$ to the value of $\delta_2 w_1$ for the case considered in 6·18, while the solution of the canonical equations of 6·15 contributes $\frac{9}{8} j_1 B^2 \sin 2N$, the sum of these giving the result 6·18 (1).

Incidentally, this exercise furnishes a reason for not continuing the solution by the Delaunay method. There are two portions to calculate instead of one, and each of these portions is large compared with their sum.

6·20. *More accurate determination of second order terms.*

The calculation of the portions of $\delta_2 c_i$, $\delta_2 w_i$ which have the small divisor ν^3 in the terms with arguments $2N$ or $N \pm \overline{N}$, is not difficult. In the equations 6·17 (5), 6·17 (6) we need to use only the portion of δw_1 which has the divisor ν^2. Thus the derivatives of R needed are

$$\frac{\partial}{\partial w_1}\left(\frac{\partial R}{\partial w_i}\right), \quad \frac{\partial}{\partial w_1}\left(\frac{\partial R}{\partial c_i}\right). \quad \dots\dots\dots\dots\dots\dots(1)$$

Now the derivatives of R with respect to w_i, c_i will have been obtained in finding the first approximation, and the derivatives (1) can be written down at once, even after numerical values have been inserted.

To calculate $\delta_2 w_1$, we need the full value of δw_1 and the values of δc_i, δw_2, δw_3 for the terms in the first line of 6·17 (7); in the second line of the latter equation, we need the principal part of δw_1 only and we can neglect δc_i, δw_2, δw_3.

In 6·17 (5), (6) we can also neglect δc_i, δw_2, δw_3 and use only the principal part of δw_1. Thus, in all cases, the terms divided by the cube of the small divisor can be obtained with the second derivatives (1) only.

6·21. *Second order secular effects.*

These are produced by inserting the values 6·17 (13) in the equations (5), (6), (7) of 6·17. It will be noted that they give d'Alembert series in the same sense as the periodic portions discussed in the same section.

There are three classes of terms present in the right-hand members of the equations to be considered.

(*a*) Terms of the form t multiplied by terms in which $j_1 = j_1' = 0$.

(*b*) Terms of the form t multiplied by terms in which $\nu \neq 0$.

(*c*) Terms in which $j_1 = j_1' = 0$ which arise when the periodic portions of δc_i, δw_i are substituted, these portions having been laid aside in the previous sections.

The integration of the equations with the terms of class (*a*) gives terms factored by t^2 and by m'^2 since constant values may be substituted for the elements in the right-hand members. It is to be noticed that $\delta_2 c_1$ contains no such terms because when $j_1 = 0$, $\partial R/\partial w_1$ and its derivatives are zero. The same result is true of the terms arising in the first line of 6·17 (7). A constant part arises from the terms of the second line which gives a term factored by t^2 in $\delta_2 w_1$. All these portions are very small since no small divisors enter.

The terms of class (*b*) give rise to differential equations of the form

$$\frac{dx}{dt} = tk \cos(\nu t + \nu_0),$$

an integral of which is

$$x = \frac{tk}{\nu} \sin(\nu t + \nu_0) + \frac{k}{\nu^2} \cos(\nu t + \nu_0).$$

Terms of this character arise in $\delta_2 c_i$, $\delta_2 w_2$, $\delta_2 w_3$. We have seen that terms of the second order with the small divisor ν^2 can usually be neglected, and the terms with the factor t/ν will rarely be sensible except for large values of t.

For $\delta_2 w_1$ we have an equation of the type

$$\frac{d^2x}{dt^2} = tk \cos(\nu t + \nu_0),$$

a particular integral of which is

$$x = -\frac{tk}{\nu^2}\cos(\nu t + \nu_0) + 2\frac{k}{\nu^3}\sin(\nu t + \nu_0).$$

The second of these terms is of the same order as those considered in 6·20; the first will be sensible for large values of t.

Class (c) gives rise to terms of the form λt in $\delta_2 c_i$, $\delta_2 w_2$, $\delta_2 w_3$ and to terms of the form λt^2 in $\delta_2 w_1$. A long period term in δw_1 possesses the small divisor ν^2 so that the resulting term in $\delta_2 w_1$ may become sensible as a result of integration, for large values of t. The proof that $\delta_2 c_1$ contains no such terms is furnished as follows.

Suppose that in the transformation of 6·6, we define R_t as containing *all* terms for which $\nu \neq 0$, and that instead of solving by the method of approximation adopted above, we write down the equation for c_{10} as given by 6·15 (6). It is

$$\frac{dc_{10}}{dt} = \frac{\partial}{\partial w_{10}}(R_{c0} + F_1 + F_t + F_c). \quad \text{...........(1)}$$

By definition, R_{c0} is independent of w_{10} and F_1, F_t, F_c contain no terms factored by t; as the remaining terms have the factor m'^2, we can insert constant values for c_{i0}, w_{20}, w_{30} and the value $n_{00}t+$const. for w_{10}. The right-hand member has then no constant term and consequently c_{10} has no term factored by t.

Next, the solution of the equation $c_1 = c_{10} + \partial S/\partial w_{10}$, to the second order, is

$$c_1 = c_{10} + \frac{\partial S_0}{\partial w_{10}} + \Sigma_j \frac{\partial^2 S_0}{\partial w_{10}\partial c_{j0}} \cdot \frac{\partial S_0}{\partial w_{j0}}. \quad \text{...........(2)}$$

The portion of this under the sign of summation contains the factor m'^2, and since S_0 consists wholly of periodic terms, it cannot contain any term factored by t. In the second term we can substitute the values of c_{i0}, w_{i0} to the order m'; these have the form $\beta_0 + \beta_1 t$, where β_0, β_1 are constants. Hence, $\partial S_0/\partial w_{10}$ will

contain only terms of the form $B\cos N_0$ or $Bt\cos N_0$, where the coefficient of t in N_0 is not zero, and B is a constant. Since it has been shown that c_{10} contains no term of the form $t \times$ const., it follows that c_1 has the same property. It does, however, contain terms of the form $Bt\cos N_0$ where the coefficient of t in N_0 is not zero.

Finally, the result is true for any function of c_1. For such terms can arise only from products of terms of the form $t\cos N_0$ with terms of the form $\cos N_0$; the former have the factor m'^2 while the latter have the factor m', so that the product will have the factor m'^3. In particular it is true for $a = c_1^2/\mu$ and for any function of a to the order m'^2. For remarks on the degree of importance to be attached to this well-known result see 7·29.

6·22. *General Summary.* The notation of 6·15 will now be resumed. The results in 6·17 to 6·21 give the values of c_{i0}, w_{i0} in the form

$$c_{i0} = \text{const.} + \delta c_{i0}, \quad w_{i0} = \text{const.} + \delta w_{i0}, \quad \text{.........(1)}$$

except in the case of w_{10} which takes the form

$$w_1 = n_{00}t + \text{const.} + \delta w_{10}. \quad \text{..................(2)}$$

The symbols δc_{i0}, δw_{i0} include all long period and secular perturbations as far as the order m'^2.

In putting
$$S = \Sigma \frac{K}{\nu} \sin N_0, \quad \text{....................(3)}$$

we have included in S only the terms corresponding to the short period terms $\Sigma K \cos N$ in R. For such terms we have seen in 6·7 that a sufficient approximation to the values of c_i, w_i in terms of c_{i0}, w_{i0} is, in general, given by

$$c_i = c_{i0} + \frac{\partial S_0}{\partial w_{i0}}, \quad w_i = w_{i0} - \frac{\partial S_0}{\partial c_{i0}}. \quad \text{...........(4)}$$

The values of c_{i0}, w_{i0} given by (1), (2) are substituted in (4). Since S_0 contains the factor m', it is sufficient to use the values of c_{i0}, w_{i0} to the first order in the second terms of (4).

Thus the short period terms to the second order are found with sufficient accuracy by substituting in them the constant

values of the elements increased by their secular and long period portions.

A literal development of R is needed to obtain the first approximation in order to obtain the first derivatives of R with respect to the elements. The second derivatives of R are needed to a lower degree of accuracy, and as far as they are usually necessary for the calculation of the second approximation to the long period terms, they can be obtained from the first derivatives after numerical values have been inscribed therein.

6·23. *Integration by parts.*

A method of integration which can be applied to non-canonical as well as to canonical equations for the variations of the elements depends on the identity

$$P \cos N = \frac{d}{dt}\left(\frac{P}{\dot{N}} \sin N\right) - \sin N \frac{d}{dt}\left(\frac{P}{\dot{N}}\right), \quad \ldots\ldots(1)$$

where $\dot{N}$ is written for dN/dt.

Suppose that two of the variables chosen be w_1 and a (or any function of a) and define n by $n^2a^3 = \mu$. Let the remaining variables be any functions of a, e, ϖ, Γ, θ which do not contain t explicitly. The equation for w_1 has the form

$$\frac{dw_1}{dt} = n + \Sigma P \cos N + Q, \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

and the equation for any one of the other variables, including n, has the form

$$\frac{dx}{dt} = \Sigma P \cos N + Q, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

where P, Q have the factor m' in both cases. Hence P, Q may be functions of any of the elements except w_1, and w_1, t will be present only in N, and in the form $j_1w_1 + j_1'w_1'$, where $w_1' = n't + \epsilon'$.

It follows that $\dot{N}$ has the form

$$j_1 n + j_1' n' + \Sigma P \cos N + Q$$

and that $\ddot{N}$ has the factor m' and the form $P \cos N + Q$. Since $\cos(N - 90°) = \sin N$, this statement includes terms of the form $\Sigma P \sin N + Q$.

Integrating (3) by the aid of (1) we have

$$x = x_0 + \Sigma \frac{P}{\dot{N}} \sin N - \Sigma \int \sin N \frac{d}{dt}\left(\frac{P}{\dot{N}}\right) dt + \int Q\, dt, \quad \ldots(4)$$

where x_0 is a constant. Since P has the factor m' and since the derivatives of all the elements present in P, $\dot{N}$ have the same factor, the third term has the factor m'^2.

In a first approximation, terms factored by m'^2 are neglected and constant values are substituted for the elements in the terms factored by m'. Hence, the first approximation to the integral of (3) is

$$x = x_0 + \Sigma \frac{P_0}{\dot{N}_0} \sin N_0 + Q_0 t. \quad \ldots\ldots\ldots\ldots\ldots(5)$$

For a second approximation, the values (5) are substituted in the second and fourth terms of (4): in the third term constant values of the elements can be used. The integrations may then be carried out in the usual manner.

The first approximation to w_1 is obtained from (2) after the substitution for n of its first approximation obtained from (5); in this approximation a term $Q_0 t$ is not present in (2). The second approximation is made in a manner similar to those outlined for the other elements.

6·24. *The case of a single long period term.*

Whenever it is possible to limit the long period terms to a single value of $j_1 w + j_1' w_1'$ and its multiples it is possible by a change of variables to eliminate the time from the Hamiltonian function, H. This function equated to a constant then constitutes an integral of the equations, and by means of this integral c_1 may be expressed as a function of the other variables and thus eliminated from the equations. The manner in which effective use can be made of this elimination is shown in Chapter VIII which treats of resonance but which is equally applicable to terms of long period.

6·25. The theory outlined in this chapter, in common with all theories which depend on the method of the variation of the elements, has a simplicity of analytical form which makes it attractive for many theoretical investiga-

tions and particularly for those which are concerned with the phenomena of resonance. But it is doubtful whether it lends itself most conveniently for the calculation of ordinary planetary perturbations. It appears, in general, to demand more extensive calculation to secure a given degree of accuracy than those methods in which the perturbations of the coordinates are obtained directly.

The chief objection is the necessity for expanding the disturbing function literally in order that the derivatives with respect to the elements of the disturbed planet may be obtained: there are six of these derivatives to be found, as against three functions to be calculated when the forces are used. A second objection is the necessity for carrying the expansion in powers of the eccentricity of the disturbed planet to one order higher than that needed in the coordinates. A third objection is the slow convergence along powers of e^2, e'^2, Γ of the series which gives the coefficient of any periodic term, especially for those terms which contain high multiples of g, g'. To a large extent this slow convergence disappears where numerical values for these elements are used at the outset of the work, particularly if the developments are made by harmonic analysis in the manner outlined in 3·17. The chief exception to these statements is the theory of the Trojan group, but this theory is so different from that of the ordinary planetary theory that comparisons are not useful. It is possibly true that all the actual cases of resonance or of very near resonance can be treated effectively by this method, but some rather extensive comparisons would be needed before any reliable statement could be made in this respect.

The literature on the subject of the application of the method of the variation of the elements to the planetary problem is extensive. The reader is referred to the standard treatises, particularly to that of Tisserand and to the articles in the *Ency. Math. Wiss.* for the earlier literature. For the later work, references and abstracts will be found in the mathematical and astronomical publications which summarise the literature annually.

6·26. Throughout this chapter it has been supposed that the mutual perturbations of two planets can be separated, so that in determining the motion of one planet that of the other can be supposed to be known. As long as we confine our attention to perturbations which depend only on the first power of the ratio of the mass of any planet to that of the sun, this procedure is justified by the fact that the coordinates of the disturbing planet only appear in a function which has the mass of this planet as a factor. Hence, any perturbations of these coordinates will produce perturbations depending on the squares or products of two disturbing masses. When we begin to calculate these higher approximations, it is evidently necessary to calculate previously the perturbations depending on the first powers of the masses for both planets.

But the general problem of three bodies admits of four integrals in addition to those arising from the uniform motion of the centre of mass, namely, the three integrals of areas or of angular momenta and the energy integral. No use has been made of these in the theory developed above and the question naturally arises as to whether they can be effectively utilised for the abbreviation of the work. In asteroid problems where there is a very small mass disturbed by a very large one, the effect of the former on the latter is negligible, and the integrals consist chiefly of portions depending on the large mass, the effect of the portions depending on the small mass being relatively small. Thus the integrals are not useful in such cases. But when the two planets have masses of the same order of magnitude, as, for example, in the case of the mutual perturbations of Jupiter and Saturn, the variations of the coordinates in the portions of the integrals due to the two planets have the same order of magnitude, and it would seem that this fact should be utilised to abbreviate the calculations. It generally appears, however, that the lack of symmetry which their use introduces, causes additional difficulties in the calculations. The more useful procedure is that of following the usual method for each of the planets and later making the integrals serve as tests of the numerical work. These tests are particularly valuable for the coefficients of any terms of very long period which may be present.

For theoretical work in the general problem of three bodies, these integrals have been much discussed. Since there are four of them, the system of variables, namely six for each planet, can be reduced from the twelfth order to the eighth.

We shall see in a later chapter that it is not always possible to proceed by following the process described at the beginning of this article. It breaks down in certain cases of resonance and notably in the case of the Trojan group. If, for example, we attempt to determine the action of Saturn on a member of this group without taking into account at the same time the action of Jupiter, quite erroneous results will be obtained. A difficulty of a similar nature occurs in dealing with the motion of a satellite disturbed by a planet other than that about which it is circulating: it is necessary to take account of the disturbing action of the sun during the computation of the disturbance caused by the planet.

CHAPTER VII

PLANETARY THEORY IN TERMS OF THE ORBITAL TRUE LONGITUDE

A. EQUATIONS OF MOTION AND METHOD FOR SOLUTION

7·1. *The equations of motion* have been derived in 1·27. The independent variable v is the longitude reckoned from a departure point within the osculating plane, while the longitude v is reckoned in the usual way from an origin in the plane of reference to the node and then along the osculating plane. The radius vector is r and i, θ are the inclination and longitude of the node of the osculating plane. The force-function is

$$\mu/r + \mu R, \qquad \text{.........................(1)}$$

so that μR now denotes the disturbing function. The remaining definitions and the equations of motion are as follows:

$$u = \frac{1}{r}, \quad \left(\frac{\mu}{q}\right)^{\frac{1}{2}} = r^2 \frac{d\mathrm{v}}{dt}, \quad \Gamma = 1 - \cos i, \qquad \text{.........(2)}$$

$$\frac{d^2 u}{d\mathrm{v}^2} + u - q = q\frac{\partial R}{\partial u} - \frac{q}{u^2}\frac{\partial R}{\partial v}\frac{du}{d\mathrm{v}} = q\frac{\partial R}{\partial u} + \frac{1}{2q}\frac{dq}{d\mathrm{v}}\frac{du}{d\mathrm{v}}, \qquad \text{...(3)}$$

$$\frac{dq}{d\mathrm{v}} = -\frac{2q^2}{u^2}\frac{\partial R}{\partial v}, \quad \frac{dt}{d\mathrm{v}} = \left(\frac{q}{\mu}\right)^{\frac{1}{2}}\frac{1}{u^2}, \quad \frac{dv}{d\mathrm{v}} = 1 + \Gamma\frac{d\theta}{d\mathrm{v}}, \qquad \text{......(4), (5), (6)}$$

$$\frac{d}{d\mathrm{v}}(\Gamma q^{-\frac{1}{2}}) = -\frac{q^{\frac{1}{2}}}{u^2}\frac{\partial R}{\partial \theta}, \quad \frac{d\theta}{d\mathrm{v}} = \frac{q}{u^2}\frac{\partial R}{\partial \Gamma}. \qquad \text{...(7), (8)}$$

The latitude L, defined by

$$\sin L = \sin i \sin(\mathrm{v} - \theta), \qquad \text{.................(9)}$$

may be found directly by solving the equation

$$\left(\frac{d^2}{d\mathrm{v}^2} + 1\right)\sin L = \frac{\sin i \cos i}{\sin(v - \theta)}\frac{q}{u^2}\frac{\partial R}{\partial \Gamma}. \qquad \text{......(10)}$$

The values of the variables u, v, t, L (or Γ, θ) are to be deduced from these equations in terms of v.

7·2. It was pointed out in 1·27 that any substitutions of the form

$$q=f(q_1), \quad u=u_1\psi(q_1)$$

leave the essential characteristics of these equations unchanged, namely, that they shall be integrable like linear equations with constant coefficients when the right-hand members have been expressed in terms of v. A transformation which renders the equations useful for the treatment of the satellite problem is

$$q=q_1^{\frac{4}{3}}, \quad u=u_1 q_1^{\frac{1}{3}}.$$

The transformation to the new variables u_1, q_1 is straightforward. For the u_1 equation we have, if D be written for d/dv,

$$D^2 u=q_1^{\frac{1}{3}}\left\{D^2 u_1+\tfrac{2}{3}Du_1\frac{Dq_1}{q_1}+\tfrac{1}{3}u_1 D\left(\frac{Dq_1}{q_1}\right)+\tfrac{1}{9}u_1\left(\frac{Dq_1}{q_1}\right)^2\right\},$$

$$\tfrac{1}{2}Du\,.\,\frac{Dq}{q}=q_1^{\frac{1}{3}}\left\{\tfrac{2}{3}Du_1\frac{Dq_1}{q_1}+\tfrac{2}{9}u_1\left(\frac{Dq_1}{q_1}\right)^2\right\},$$

so that Du_1 disappears. This and the remaining equations become, if $-2R_v'$ be put for $u^3\partial R/\partial u$, and R_v for $u^2 R$,

$$D^2 u_1+u_1-q_1=-\frac{2}{u_1^3}R_v'-u_1\left\{\tfrac{1}{3}D\left(\frac{Dq_1}{q_1}\right)-\frac{1}{9}\left(\frac{Dq_1}{q_1}\right)^2\right\},$$

$$\frac{Dq_1}{q_1}=-\frac{3}{2}\frac{1}{u_1^4}\frac{\partial R_v}{\partial v}, \quad Dt=\frac{1}{\sqrt{\mu}}\frac{1}{u_1^2}, \quad Dv=1+\Gamma D\theta,$$

$$\frac{d\Gamma}{\Gamma}=\frac{2}{3}\frac{Dq_1}{q_1}-\frac{1}{u_1^4}\frac{1}{\Gamma}\frac{\partial R_v}{\partial\theta}, \qquad D\theta=\frac{1}{u_1^4}\frac{\partial R_v}{\partial\Gamma}.$$

When the ratio of the distances is neglected we have $R_v'=R_v$ and each is independent of u_1. This portion constitutes the chief part of the disturbing function in the satellite problem.

It will be noticed that Dt is a function of u_1 only, and in fact that the substitution $u_1=(Dt)^{-\frac{1}{2}}\mu^{-\frac{1}{4}}$ eliminates the radius vector from the equations of motion. No particular advantage, however, appears to be gained by this elimination.

7·3. *The method for solution.*

The equations will be solved by continued approximation. The function R contains as a factor m'/μ, the ratio of the mass of the disturbing planet to the sum of the masses of the sun and the disturbed planet. This factor being always small (its maximum value is less than ·001), the first step is the solution of the equa-

tions with $R=0$. As we have seen in Chap. III, this solution gives elliptic motion and the solution will be called the *elliptic approximation*. The results consist of expressions for the variables u, t, ..., in terms of v.

For the *first approximation* to the disturbed motion, these expressions are substituted in the terms which have m'/μ as a factor and the equations are solved again. Analytically, the process presents no difficulties since, with the exception of the equation for t, the right-hand members become functions of v only, while the left-hand members are linear with constant coefficients. The value of q is first obtained and then those of u, t and of the remaining variables.

The second approximation is similarly obtained by substituting the results from the first approximation in the terms which have m'/μ as a factor and proceeding as before. It is rarely necessary to go beyond this stage in planetary problems and, in fact, a second approximation is necessary in general only for those terms which, on account of their long periods, have received large factors during the integration of the equations giving the first approximation.

The system of differential equations is one of the seventh order while that from which it was derived was of the sixth order requiring six arbitrary constants. The additional arbitrary constant necessary in the new system owing to the differential definition of v, will be defined as follows. The final expression of $v-\mathrm{v}$ in terms of v is a sum of periodic terms and powers of v: when all these terms are neglected we are to have $v=\mathrm{v}$. In general, this is equivalent to putting $v=\mathrm{v}$ when $m'=0$. But there are sometimes periodic terms present whose coefficients do not vanish with m' but whose arguments become constant when $m'=0$; the relation $v=\mathrm{v}$ is to hold when these terms are suppressed.

7·4. *The elliptic approximation.* When $R=0$ we have q, Γ, θ, i constant and with $D=d/d\mathrm{v}$,

$$(D^2+1)\,u=q, \quad (D^2+1)\sin L=0, \quad Dt=u^{-2}(q/\mu)^{\frac{1}{2}}, \quad D\mathrm{v}=1.$$

In accordance with the definition given in the previous section, the last equation gives $v = \mathrm{v}$. The solution of the equation for u can be written

$$\frac{1}{r} = u = q + qe \cos(\mathrm{v} - \varpi),$$

where e, ϖ are arbitrary constants. As we have seen in Chap. III, this is the equation of an ellipse with the origin at one focus. If $2a$, e, ϖ be the major axis, eccentricity and longitude of the nearer apse of the curve, we have

$$1/q = a(1 - e^2).$$

Following the notations of Chap. III, namely, n defined by $n^2 a^3 = \mu$, with $nt + \epsilon - \varpi$ as the mean anomaly and X as the eccentric anomaly, we obtain t expressed in terms of v by the equations [cf. 3·2 (16) and (20)],

$$nt + \epsilon - \varpi = X - e \sin X, \quad \tan \tfrac{1}{2} X = \left(\frac{1 - e}{1 + e}\right)^{\frac{1}{2}} \tan \tfrac{1}{2}(\mathrm{v} - \varpi), \qquad \ldots\ldots(1)$$

or by [cf. 3·8 (3)],

$$nt + \epsilon = \mathrm{v} - E_f, \quad E_f = 2e \sin(\mathrm{v} - \varpi) - \tfrac{3}{4} e^2 \sin 2(\mathrm{v} - \varpi) + \ldots. \qquad \ldots\ldots(2)$$

The solution of the equation for L gives

$$\sin L = \sin i \sin(\mathrm{v} - \theta).$$

The arbitrary constants of the solution are q, e, ϖ, i, θ, ϵ. It is, however, more convenient to regard n as one of the fundamental arbitraries since it is determined more directly from observation, and to regard q as a function of n, e, defined by means of the equations $1/q = a(1 - e^2)$, $n^2 a^3 = \mu$.

The adopted definitions of u, q, R give them the dimension -1 in length. If we put u/a_0, q/a_0, R/a_0 for these symbols and define n_0 by the equation $n_0^2 a_0^3 = \mu$, none of the equations except that for Dt is altered and the latter becomes $n_0 Dt = q^{\frac{3}{2}} u^{-2}$. The unit of length is at our disposal: it will be found convenient to so choose a_0 that n_0 is the mean value of the angular velocity of the disturbed body which has been adopted. With this definition therefore, we put $a_0 = a$, $n_0 = n$ in finding the first approximation to the perturbations.

We shall suppose this transformation to have been made so that

$$u = (1 + e\cos f) \div a(1 - e^2)$$

with $a = 1$ is the elliptic approximation, to the end of 7·18.

B. The First Approximation to the Perturbations

7·5. *Development of the disturbing function.*

According to the plan outlined in 7·3, the elliptic approximation, that is, the values of the coordinates in terms of v and six arbitrary constants, found in 7·4, is to be substituted in the derivatives of R which are present in the right-hand members of the equations of motion.

Since the disturbing function is here denoted by μR we have, from 1·10,

$$R = \frac{m'}{\mu}\left(\frac{1}{\Delta} - \frac{r\cos S}{r'^2}\right), \quad \Delta^2 = r^2 + r'^2 - 2rr'\cos S,$$

$$\begin{aligned}\cos S &= \cos(v - \theta)\cos(v' - \theta) + \cos I \sin(v - \theta)\sin(v' - \theta)\\ &= (1 - \tfrac{1}{2}\Gamma)\cos(v - v') + \tfrac{1}{2}\Gamma\cos(v + v' - 2\theta).\end{aligned}$$

The disturbing function contains the coordinates r', v' of the disturbing body and these must be expressed in terms of v. Since the orbit of the disturbing body is used as the plane of reference, we have $i = I$.

The work is best done in two steps. First, r', v' are expressed in terms of t by means of

$$\frac{1}{r'} = \frac{1 + e'\cos(v' - \varpi')}{a'(1 - e'^2)},$$

$$v' = n't + \epsilon' + 2e'\sin(n't + \epsilon' - \varpi') + \dots,$$

as found in 3·11, and then in terms of v by means of the relations similar to 7·4 (1), (2).

It is found convenient to introduce the angles

$$f = \mathrm{v} - \varpi, \quad f_1 = \frac{n'}{n}(\mathrm{v} - \epsilon) + \epsilon' - \varpi'. \quad \dots\dots(1), (2)$$

Evidently f is the true anomaly of the disturbed planet and f_1 is the mean value of the true anomaly of the disturbing planet when the latter is expressed in terms of v. The derivatives of f, f_1 are in the ratio $n:n'$, that is, in the ratio of the mean motions.

The disturbing function is ultimately expressed as a sum of cosines of multiples of the angles f, f_1, $\varpi - \varpi'$, $\varpi + \varpi' - 2\theta$, with coefficients which depend on a, a', e, e', Γ. The chief difference in the literal form of the expansion from that obtained with t as the independent variable is the presence of a/a' in the form of powers of n'/n as well as directly. But as these powers of n'/n occur only in rapidly converging forms they cause little additional trouble*.

The expression for R used above assumes that the plane of motion of the disturbing planet is fixed and adopted as the plane of reference. It should be pointed out that, as far as perturbations of the first order with respect to the masses are concerned, it makes no difference whether this plane is fixed or moving. For since its motions are produced solely by other disturbing bodies, they contain the disturbing masses as factors. But the effects of the disturbing body enter the equations of motion only through R which has m' as a factor: these motions will therefore produce perturbations having the product of two disturbing masses as a factor.

Hence, if we have solved the problem under the assumption that the plane of reference is fixed, the solution to the first order still holds when we transform to another plane of reference which is actually fixed, the motion of the former plane being included in the transformation. In other words, we need to take into account only the *geometrical* effect of the motion of the plane of the disturbing body and can neglect its *dynamical* effect on the disturbed body. Actually, these second order dynamical effects are, in most cases, so small that they may be neglected in making comparisons with observations.

Another point of a similar character may be mentioned. We are substituting constant values for the various elements in the expressions for the coordinates in R. To the first order of the disturbing forces it makes no theoretical difference what these elements are, whether, for example, they are osculating elements at one date or another, or are mean elements derived

* A method, similar to that just outlined, for developing the disturbing function in terms of the true longitude, is given by C. A. Shook, *Mon. Not. R.A.S.* vol. 91 (1931), p. 553. In this paper will be found the literal development to the second order with respect to the eccentricities and inclination.

in some manner, for all these sets differ from one another only by magnitudes of the order of the disturbing forces. But when we compute to the second order actual definitions are necessary. In general, we get better accuracy when we use mean elements if they are known. Such elements are best found after the theory has been completed and their insertion usually involves small corrections to those which have been used in forming the theory: in most cases such corrections are easily made.

For these reasons, it is not necessary, in forming the equations for the first approximation, to use a separate notation for the elements used in the elliptic approximation and for the new values which may be assigned to them in the first approximation to the perturbations.

7·6. *Numerical developments of the disturbing forces.* The following method of calculation is based on the possibility of expressing the disturbing function and the disturbing forces in the form

$$\Sigma_i K_i \cos i\,(f - f_1 + \varpi - \varpi') + \Sigma_i K_i' \sin i\,(f - f_1 + \varpi - \varpi'), \quad (1)$$

where the coefficients are series of the form

$$\Sigma_{j, j_1} A_{j, j_1} \cos (jf + j_1 f_1) + \Sigma_{j, j_1} A'_{j, j_1} \sin (jf + j_1 f_1), \quad \ldots(2)$$

the latter coefficients containing the numerical values of the eccentricities, of the inclination and of $2\varpi - 2\theta$. The coefficients in the series (2) are supposed to be calculated by double harmonic analysis for each value of i needed.

The principal reasons for the adoption of this plan are first, the slow convergence of the coefficients K_i, K_i' with increasing values of i, and second, the comparatively rapid convergence of the series for A_{j, j_1}, A'_{j, j_1}, so that only a few special values of f, f_1 are needed for the harmonic analysis. No additional calculation is involved by the retention of $\varpi - \varpi'$ in a literal form as far as the final step.

Omitting the factor m'/μ, we have

$$r^2 R = \frac{r^2}{\Delta} - \frac{r^3 \cos S}{r'^2}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

$$\frac{\partial R}{\partial u} = -r^2 \frac{\partial R}{\partial r} = \frac{r^3 - r^2 r' \cos S}{\Delta^3} + \frac{r^2 \cos S}{r'^2}$$

$$= \frac{1}{2}\frac{r}{\Delta} - r\frac{r'^2 - r^2}{2\Delta^3} + \frac{r^2 \cos S}{r'^2}, \quad \ldots\ldots\ldots\ldots(4)$$

$$\frac{\partial}{\partial \theta}(r^2 R) = \left(\frac{r^3}{\Delta^3} - \frac{r^3}{r'^3}\right) r' \Gamma \sin (v + v' - 2\theta), \quad \ldots\ldots(5)$$

$$\frac{\partial}{\partial \Gamma}(r^2 R) = -\left(\frac{r^3}{\Delta^3} - \frac{r^3}{r'^3}\right) r' \sin(v-\theta)\sin(v'-\theta), \quad \ldots(6)$$

$$\frac{\partial}{\partial v}(r^2 R) = \frac{\partial}{\partial \varpi}(r^2 R) = -\frac{\partial}{\partial \varpi'}(r^2 R) - \frac{\partial}{\partial \theta}(r^2 R). \quad \ldots(7)$$

When 7·1 (10) is used instead of 7·1 (7), (8), the calculation of (5) is needed only for that of (7); (6) is then calculated without the factor $\sin(v-\theta)$. Cf. 1·28.

It will appear below that of these, the development of (7) requires the highest degree of accuracy. Somewhat lower accuracy will serve for (4), and still lower for (5), (6); since (5) contains the factor Γ, these facts require that (3) shall be carried to the highest accuracy of all the functions. The most extensive part of the work is the development of the functions $r^2/\Delta, r^3/\Delta^3$, the former being needed to a higher degree of accuracy than the latter.

7·7. *Numerical development of the disturbing function.*

Define A, B, by the equations

$$\left.\begin{aligned} A\cos(v-\theta-B) &= \cos(v-\theta), \\ A\sin(v-\theta-B) &= \cos I \sin(v-\theta). \end{aligned}\right\} \quad \ldots\ldots\ldots(1)$$

The expression for Δ^2 in 7·5 can then be written

$$\Delta^2 = r^2 + r'^2 - 2rr' A\cos(v-v'-B). \quad \ldots\ldots\ldots(2)$$

From (1) we deduce

$$A^2 = 1 - \sin^2 I \sin^2(v-\theta), \quad \ldots\ldots\ldots\ldots\ldots(3)$$

$$\left.\begin{aligned} A\sin B &= \sin^2 \tfrac{1}{2} I \sin(2v-2\theta), \\ A\cos B &= 1 - 2\sin^2 \tfrac{1}{2} I \sin^2(v-\theta). \end{aligned}\right\} \quad \ldots\ldots\ldots(4)$$

Since $v = f + \varpi$, the special values of A, B corresponding to the chosen special values of f can be calculated from (3), (4) when the numerical values of Γ, $\varpi - \theta$ are given.

Next, define r_1, C by

$$(r_1^2 + r'^2)C^2 = r^2 + r'^2, \quad r_1 C^2 = rA, \quad \ldots\ldots\ldots(5)$$

so that $$\Delta^2 = C^2\{r_1^2 + r'^2 - 2r_1 r' \cos(v-v'-B)\}. \quad \ldots\ldots(6)$$

The special values of r, r' corresponding to special values of f, f_1 having been found, those of r_1, C can be obtained conveniently by calculating λ, λ_1 from

$$\tan\lambda = \frac{r}{r'}, \quad \sin 2\lambda_1 = A\sin 2\lambda, \quad \ldots\ldots\ldots\ldots(7)$$

and then r_1, C from

$$r_1 = r' \tan \lambda_1, \quad C = \frac{\cos \lambda_1}{\cos \lambda}, \quad \ldots\ldots\ldots\ldots\ldots \quad (8)$$

equations which will be found to satisfy (5).

The expression (6) for Δ^2 gives

$$\frac{r'^2}{\Delta} = \frac{r'^2}{Cr'}\left\{1 + \frac{r_1^2}{r'^2} - 2\frac{r_1}{r'}\cos(v - v' - B)\right\}^{-\frac{1}{2}}, \quad \ldots\ldots(9)$$

$$\frac{r'^3}{\Delta^3} = \frac{r'^3}{C^3 r'^3}\left\{1 + \frac{r_1^2}{r'^2} - 2\frac{r_1}{r'}\cos(v - v' - B)\right\}^{-\frac{3}{2}}. \quad \ldots(10)$$

Methods for the expansions of these functions have been given in 4·23 and the following sections. The particular form which is useful here is that in which we put

$$(1 + \alpha^2 - 2\alpha \cos \psi)^{-s} = \beta_s^{(0)} + \Sigma\, 2\beta_s^{(i)} \cos i\psi, \quad \ldots(11)$$

with $\alpha = r_1/r'$, $\psi = v - v' - B$, $i = 1, 2, \ldots$, $s = \frac{1}{2}, \frac{3}{2}$.

The expressions for the $\beta_s^{(i)}$ are given by 4·24 (6) with $\kappa = 1$, $\alpha_1 = \alpha$, namely,

$$\beta_{\frac{1}{2}}^{(i)} = \frac{\alpha^i}{(1 - \alpha^2)^{\frac{1}{2}}} g_{\frac{1}{2}}^{(i)}, \quad \beta_{\frac{3}{2}}^{(i)} = \frac{\alpha^i}{(1 - \alpha^2)^{\frac{3}{2}}} g_{\frac{3}{2}}^{(i)},$$

where
$$g_{\frac{1}{2}}^{(i)} = \frac{1 . 3 \ldots (2i - 1)}{2 . 4 \ldots 2i} F(\tfrac{1}{2}, \tfrac{1}{2}, i + 1, -p),$$

$$g_{\frac{3}{2}}^{(i)} = \frac{3 . 5 \ldots (2i + 1)}{2 . 4 \ldots 2i} F(-\tfrac{1}{2}, \tfrac{3}{2}, i + 1, -p),$$

in which $p = \alpha^2/(1 - \alpha^2)$.

The methods developed for the calculation of these functions depend on the numerical value of α being given. In the present case these numerical values are the special values of r_1/r'. The efficiency of the method outlined here depends on the existence of tables giving the coefficients for different values of α*.

* The tables of Brown and Brouwer, *l.c.* p. 103, give $\log 2g_s^{(i)}$ for $i = 0$ to 11 and for $s = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}$ to 8 places of decimals and for $s = \frac{7}{2}$ to 7 places of decimals. They are tabulated for values of p at intervals of ·01 up to $p = 2·5$ or $\alpha = ·83$. For higher values of p up to $p = 6$ ($\alpha = ·93$) rapidly converging series are given. A separate table for $\alpha = ·90$ to ·95 is added.

In this way, each of the functions 7·6 (3), (4), (5), (6) is expanded into a series having the form

$$\Sigma\, 2C^{(i)} \cos i\,(v - v' - B), \quad \ldots\ldots\ldots\ldots\ldots(12)$$

in which the special values of the coefficients for each required value of i have been obtained.

We next put

$$v - v' - B = f + \varpi - f_1 - \varpi' - B_1,$$

where $$B_1 = f' - f_1 + B,$$

and calculate the special values of B_1.

The final special values to be computed are those of

$$C^{(i)} \cos iB_1, \quad C^{(i)} \sin iB_1,$$

for each value of i. Each of these functions is then analysed into a series of the form

$$\Sigma_{j, j_1} L_{j, j_1} \cos(jf + j_1 f_1) + \Sigma_{j, j_1} L'_{j, j_1} \sin(jf + j_1 f_1). \quad \ldots(13)$$

The results give series having the form 7·6 (1), (2). After the derivative of 7·6 (3) with respect to ϖ' has been formed, the numerical value of $\varpi - \varpi'$ is inserted in all of them, the various terms having the same multiples of f, f_1 in their arguments are collected and each expression is put into the form (13) or, if desired, into the form

$$M_{j, j_1} \cos(jf + j_1 f_1 - N_{j, j_1}). \quad \ldots\ldots\ldots\ldots(14)$$

The most important preliminary step is the expression of f' in terms of f_1, f.

7·7a. *Expression of the true anomaly of the disturbing planet in terms of that of the disturbed planet.*

When harmonic analysis is to be used (App. A), the following method gives the required transformation rapidly.

We have, according to previous definitions,

$$g = f - E_f, \quad g' = f_1 - \frac{n'}{n} E_f = f_1 - \delta f_1,$$

where E_f is the equation of the centre with eccentricity e and true anomaly f. Also

$$f' = g' + E_{g'} = f_1 - \delta f_1 + E(f_1 - \delta f_1),$$

where $E_{g'} = E(f_1 - \delta f_1)$ is the equation of the centre of the disturbing planet with eccentricity e' and mean anomaly $g' = f_1 - \delta f_1$. If this function be expanded in powers of δf_1 we obtain, if $E'(f_1)$ denotes $dE(f_1)/df_1$,

$$f' = f_1 + E(f_1) - \{1 + E'(f_1)\}\,\delta f_1 + \frac{1}{2!}\frac{d}{df_1}E'(f_1)\,.\,(\delta f_1)^2 - \frac{1}{3!}\frac{d^2}{df_1^2}E'(f_1)\,.\,(\delta f_1)^3 + \dots.$$

Since $f_1 + E(f_1) = \bar{f}$ is the expression for a true anomaly in which e' is the eccentricity and f_1 takes the place of the mean anomaly, we can write the equation in the form

$$f' = \bar{f} - \frac{d\bar{f}}{df_1}\left(\frac{n'}{n}E_f\right) + \frac{1}{2!}\frac{d^2\bar{f}}{df_1^2}\left(\frac{n'}{n}E_f\right)^2 - \dots.$$

The special values of E_f are calculated with special values of f in the usual manner. Those of $\bar{f}$ with special values of f_1 are similarly obtained. For the derivatives we have

$$\frac{d\bar{f}}{df_1} = \frac{(1 + e'\cos\bar{f})^2}{(1 - e'^2)^{\frac{3}{2}}}.$$

Hence

$$\left(\frac{d\bar{f}}{df_1}\right)^{\frac{1}{2}} = (1 - e'^2)^{-\frac{3}{4}}(1 + e'\cos\bar{f}).$$

If we denote the successive derivatives of $\bar{f}$ with respect to f_1 by the notations $\dot{f}, \ddot{f}, \dots$, and put $e_1 = e'(1 - e'^2)^{-\frac{3}{4}}$, we can obtain the following formulae for their successive determination:

$$\tfrac{1}{2}\ddot{f} = -e_1\dot{f}^{\frac{3}{2}}\sin\bar{f}, \quad \tfrac{1}{6}\dddot{f} = \frac{1}{\dot{f}}(\tfrac{1}{2}\ddot{f})^2 - \tfrac{1}{3}\dot{f}^{\frac{5}{2}}e_1\cos\bar{f},$$

$$\frac{1}{24}f^{\text{iv}} = \frac{11}{4}\frac{\dddot{f}}{6}\cdot\frac{\ddot{f}}{2}\cdot\frac{1}{\dot{f}} - \frac{7}{4}\left(\frac{\ddot{f}}{2}\right)^3\frac{1}{\dot{f}^2} + \frac{e_1}{12}\dot{f}^{\frac{7}{2}}\sin\bar{f},$$

which will be sufficient for all practical needs.

The calculation of $\dddot{f}, f^{\text{iv}}$ from the series 3·16 (2) will be found to be sufficiently accurate in most cases and is rather easier than that from the formulae just given.

The amount of calculation needed in any particular case depends on a variety of circumstances. Before undertaking calculation on the general plan outlined above, a preliminary survey should be made to find the

order of magnitude of the term with the largest coefficient in the longitude, or, in the present case, in t expressed in terms of the longitude. Usually, the term is one having a long period. The order of magnitude with respect to the eccentricities and inclination for a term with argument $jf+j_1f_1$ is $|j+j_1|$. While a rough approximation to the coefficient can be obtained by following the method developed in 7·38 below, the degree of accuracy, that is, the number of places of decimals needed in the calculation, can be found from the number of significant figures needed to obtain this coefficient with the required accuracy.

The accuracy possible with the methods developed above is theoretically unlimited, but is practically limited by the accuracy of the tables of the coefficients $g_s^{(i)}$. Those referred to in the footnote on p. 178 are sufficient to obtain solutions of practically all the planetary problems in the solar system with the accuracy needed at the present time; the determination of the great inequality in the motion of Saturn is probably the limit in this respect.

It may be pointed out that since $A \gtreqless 1$, $C \geqq 1$ (p. 178), the inclusion of the inclination in r_1/r' in general tends to diminish this ratio and therefore to increase the rate of convergence. Thus, if we can obtain a certain degree of accuracy with $I=0$, we can obtain at least the same degree of accuracy with $I \neq 0$. The method is thus particularly effective for large inclinations.

The method of procedure outlined above is a general one. The experienced computer will see various ways in which it may be abbreviated. One important choice is the number of special values to be adopted for f, f_1. In the majority of minor planet problems, the values of f, f_1 at intervals of 45° will serve. If the eccentricity of the planet or the inclination is large, additional values at intervals of 60° may be used: these additional values merely involve corrections of the coefficients in the last part of the process —that of the harmonic analysis—so that all the previous work is fully utilised. It is not difficult to settle at the outset the number of places of decimals required, but it is not easy to say how many special values should be used. The work can be started with the minimum number and others can be added afterwards without the loss of the previous calculations.

7·8. *Solution of the equations.*

The methods of the previous sections give the expression of R in the form

$$R = \Sigma K \cos(jf + j_1 f_1 + k), \qquad j = 0, 1, 2, \ldots;\ j_1 = 0, \pm 1, \pm 2, \ldots.$$

When a literal development is made the angles k are multiples of $\varpi - \varpi'$, $\varpi + \varpi' - 2\theta$, and the coefficients K are functions of a/a', e, e', Γ. In a numerical development the terms having the

same values of j, j_1 are gathered together and R and its derivatives are expressed in the form

$$\beta_0 + \Sigma\,\beta \cos(jf + j_1 f_1) + \Sigma\beta' \sin(jf + j_1 f_1),$$

or in the form

$$\beta_0 + \Sigma\bar{B} \cos(jf + j_1 f_1 - \bar{B}_1),$$

where β_0, β, β', $\bar{B}$, $\bar{B}_1$ are numerical quantities dependent on all the elements; in these expressions the terms in which $j = j_1 = 0$ are gathered into the symbol β_0.

The constant β_0 is independent of v and is implicitly a function of the angles $\varpi - \varpi'$, $\varpi + \varpi' - 2\theta$. The terms present in this constant when expressed in a literal form possess the property (6·4) associating a power of e in the coefficient with the same multiple of ϖ in the angle, with similar properties for ϖ', e' and for 2θ, Γ. The corresponding properties are obtained when j, j_1 are not both zero by putting $\varpi = v - f$, $\varpi' = v_1 - f_1$ and associating powers of e, e', 2θ with the respective multiples of f, f_1, 2θ—a statement which is easily seen to be true by referring back to the development of R in 4·14.

Finally, since we have put $v = \mathrm{v}$, we have

$$D(jf + j_1 f_1) = j + j_1 n'/n = s,$$

so that s becomes a divisor of the coefficient when we integrate one of these expressions.

It is evident that $Rf(r, r')$ will possess these same properties.

7·9. *The equation for q.* This equation has the form

$$Dq = \beta_0 + \Sigma\,\beta \cos(jf + j_1 f_1) + \Sigma\,\beta' \sin(jf + j_1 f_1), \quad \ldots\ldots(1)$$

and its integral is

$$q = q_0 + \beta_0 \mathrm{v} + \Sigma\frac{\beta}{s} \sin(jf + j_1 f_1) - \Sigma\frac{\beta'}{s} \cos(jf + j_1 f_1), \quad \ldots(2)$$

where q_0 is an arbitrary constant to be defined later. The term $\beta_0 \mathrm{v}$ is the secular part of q. The terms for which s is small compared with unity are those of long period, the remaining terms having 'short' periods, that is, periods of the same order of magnitude as $2\pi/n$, the period of revolution of the planet, and shorter.

7·10. *The equation for* u. The right-hand member has already been developed into an expansion of the form 7·9 (1). In the left-hand member, the value of q just obtained is substituted. The periodic terms in q are added to the terms of the right-hand member and the equation takes the form

$$D^2 u + u = q_0 + \beta_0 \mathrm{v} + \beta_1 \cos f + \beta_1' \sin f$$
$$+ \beta_{u0} + \Sigma \beta_u \cos (jf + j_1 f_1) + \Sigma \beta_u' \sin (jf + j_1 f_1),$$

the terms for which $j_1 = 0, j = 0, 1$, simultaneously, being isolated.

The integral of this equation is

$$u = q_0 + \beta_0 \mathrm{v} + \tfrac{1}{2}\beta_1 \mathrm{v} \sin f - \tfrac{1}{2}\beta_1' \mathrm{v} \cos f + e_c \cos f + e_s \sin f + u_p, \qquad \text{......(1)}$$

where

$$u_p = \beta_{u0} + \Sigma \frac{\beta_u}{1-s^2} \cos (jf + j_1 f_1) + \Sigma \frac{\beta_u'}{1-s^2} \sin (jf + j_1 f_1),$$

as can be seen by submitting each member to the operator $D^2 + 1$; e_c, e_s are the arbitrary constants in the solution.

Now after the substitution u/a for u, q/a for q, the elliptic values of u, q are

$$u = \frac{1 + e \cos f}{1 - e^2}, \quad q = \frac{1}{1 - e^2}, \quad f = \mathrm{v} - \varpi,$$

with e, ϖ as arbitrary constants. As q_0, e_c, e_s are at our disposal we can put

$$e_c = \frac{e}{1 - e^2}, \quad e_s = 0, \quad q_0 = \frac{1}{1 - e^2},$$

so that the remaining terms would constitute the perturbations. These values of e_c, e_s will be adopted, but instead of that for q_0 we shall put

$$q_0 = \frac{1}{1 - e^2} + \delta q_0,$$

where δq_0 is still arbitrary. It will later be defined to be such that the mean value of $n\,Dt$ shall be unity.

We shall next show that u, q may be written in the forms

$$u = u_0 + \delta q_0 + u_p, \quad q = 1/(1 - e^2) + \delta q_0 + q_p, \qquad \text{......(2)}$$

where $$u_0 = \frac{1 + e\cos(f - \varpi_1 \mathrm{v})}{1 - e^2}, \quad e = e_0 + e_1 \mathrm{v}. \qquad \ldots\ldots(3)$$

Here e_0 is the value of e which has been used in the development of R; ϖ_1, e_1 are small constants whose squares may be neglected.

The expansion of u_0 in powers of e_1, ϖ_1, gives

$$u_0 = \frac{1+e_0\cos f}{1-e_0^2} + \frac{2e_0 e_1}{(1-e_0^2)^2}\mathrm{v} + \frac{1+e_0^2}{(1-e_0^2)^2} e_1 \mathrm{v} \cos f + \frac{e_0}{1-e_0^2}\varpi_1 \mathrm{v} \sin f.$$

Comparison of this with (1) shows that if we put

$$e_1 = -\tfrac{1}{2}\frac{(1-e_0^2)^2}{1+e_0^2}\beta_1', \quad \varpi_1 = \tfrac{1}{2}\frac{1-e_0^2}{e_0}\beta_1,$$

the coefficients of $\mathrm{v}\cos f$, $\mathrm{v}\sin f$ in (1) will be included in u_0. The term $\beta_0 \mathrm{v}$ will also be included if

$$\beta_0 = \frac{2e_0 e_1}{(1-e_0^2)^2} = -\frac{e_0\beta_1'}{1+e_0^2}.$$

The argument in 7·24 shows that this relation is satisfied, so that it constitutes a useful test of the accuracy of a part of the numerical calculations. Hence u and q have the form (2).

The terms $e_1\mathrm{v}$, $\varpi_1\mathrm{v}$ are called the secular motions of the eccentricity and longitude of perihelion. Expressed in time they would be $e_1 nt$, $\varpi_1 nt$.

Since we are neglecting squares of the disturbing force we may insert these secular parts in the perturbations u_p, q_p. If a development in which the literal values of e, ϖ have been retained is made, this can be done by replacing f by $f - \varpi_1\mathrm{v}$ and ϖ by $\varpi + \varpi_1\mathrm{v}$, and e by $e + e_1\mathrm{v}$ in u_p, q_p—a procedure which is advantageous as will appear later. Usually, however, it is not possible to make these changes because it is customary to use numerical developments in order to save labour.

7·11. *The equation for t.* Since δq_0, u_p, u_q are of the order of the disturbing forces, we have, as far as the first order,

$$nDt = \frac{q^{\frac{1}{2}}}{u^2} = \frac{(1-e^2)^{\frac{3}{2}}}{\{1 + e\cos(f - \varpi_1\mathrm{v})\}^2} \times \left\{1 + \tfrac{1}{2}(1-e^2)(\delta q_0 + q_p) - 2\frac{\delta q_0 + u_p}{u_0}\right\}.$$

In the terms factored by δq_0, q_p, u_p we can put $e_1 = 0$, $\varpi_1 = 0$, since the products would be of the second order. Hence

$$nDt = \frac{(1-e^2)^{\frac{3}{2}}}{\{1+e\cos(f-\varpi_1 \mathrm{v})\}^2} + \tfrac{1}{2}\frac{(1-e_0^2)^{\frac{3}{2}}}{(1+e_0\cos f)^2}(\delta q_0 + q_p)$$
$$- 2\frac{(1-e_0^2)^{\frac{5}{2}}}{(1+e_0\cos f)^3}(\delta q_0 + u_p). \quad \ldots\ldots(1)$$

We now determine δq_0 to be such that in the expansion of the right-hand member as a sum of periodic terms, the constant term shall be unity.

By 3·8 (3), the constant part of the first term is 1. It will be shown in 7·13 that if we expand $(1+e_0\cos f)^{-3}$ into a Fourier series the constant term is $(1+\frac{1}{2}e_0^2)(1-e_0^2)^{-\frac{5}{2}}$. The constant term in the coefficient of δq_0 is therefore

$$\tfrac{1}{2}(1-e_0^2) - (2+e_0^2) = -\tfrac{3}{2}(1+e_0^2).$$

The required condition therefore gives

$$\delta q_0 = \tfrac{2}{3}\frac{1}{1+e_0^2} \times \text{constant term in the expansion of}$$
$$\tfrac{1}{2}\frac{(1-e_0^2)^{\frac{5}{2}}}{(1+e_0\cos f)^2}q_p - 2\frac{(1-e_0^2)^{\frac{5}{2}}}{(1+e_0\cos f)^3}u_p.$$

It is evident that we only need those portions of r_p, u_p which are independent of the argument f_1. It is recalled that q_p contains no constant term and that u_p contains no term with argument f.

When δq_0 has been found, its value is inserted in (1). The second and third terms are then calculated in the form

$$\frac{(1-e_0^2)^{\frac{5}{2}}}{(1+e_0\cos f)^3}\{(1+e_0\cos f)(\tfrac{1}{2}\delta q_0 + \tfrac{1}{2}q_p) - 2\delta q_0 - 2u_p\},$$

the constant term of which should vanish, so that these portions give a sum of periodic terms of the form

$$\Sigma\beta_t\cos(jf + j_1f_1) + \Sigma\beta_t'\sin(jf + j_1f_1).$$

7·12. *Integration of the equation for t.* If we neglect all perturbations so that $e = e_0$, $\varpi_1 = 0$, we obtain by 3·8 (3)

$$nt + \epsilon = \mathrm{v} - E_f, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

where E_f is the equation of the centre expressed in terms of the true anomaly f. To obtain the integral when e_1, ϖ_1 are not neglected put $e_0 + e_1 \mathrm{v}$, $f - \varpi_1 \mathrm{v}$ for e, f in (1) and differentiate. We obtain, if squares of e_1, ϖ_1 are neglected,

$$nDt = 1 - \frac{\partial E_f}{\partial f} - e_1 \frac{\partial E_f}{\partial e} + \varpi_1 \frac{\partial E_f}{\partial f}.$$

The first two terms of the right-hand member evidently give

$$(1 - e^2)^{\frac{3}{2}} \{1 + e \cos (f - \varpi_1 \mathrm{v})\}^{-2};$$

in the third and fourth terms we can put $e = e_0$, $\varpi_1 \mathrm{v} = 0$. Hence the integral of the first term of 7·11 (1) gives

$$nt + \epsilon = \mathrm{v} - E_f - e_1 \int \frac{\partial E_f}{\partial e} df + \varpi_1 \int \frac{\partial E_f}{\partial f} df,$$

where ϵ is the constant of integration, and the values $e_0 + e_1 \mathrm{v}$ and $f - \varpi_1 \mathrm{v}$ are used for the eccentricity and true anomaly in the expression for the second term, that is, for the equation of the centre. The integral of the third term is given by the formula 7·13 (4) below; that of the fourth term is $\varpi_1 E_f$.

The remaining terms in the expression 7·11 (1) for nDt, the form of which is given at the end of 7·11, are integrated immediately. We obtain

$$nt + \epsilon = \mathrm{v} - E_f(1 - \varpi_1) - e_1 F_f$$
$$+ \Sigma \frac{\beta_t}{s} \sin (jf + j_1 f_1) - \Sigma \frac{\beta_t'}{s} \cos (jf + j_1 f_1),$$

where $s = j + j_1 n'/n$, E_f is the equation of the centre with eccentricity $e_0 + e_1 \mathrm{v}$ and true anomaly $f - \varpi_1 \mathrm{v}$ and, by 7·13 (4),

$$F_f = \frac{1}{e_0} (1 - e_0^2)^{\frac{1}{2}} \left\{ \frac{1}{1 + e_0 \cos f} - 1 - \log (1 + e_0 \cos f) \right\}$$
$$= \sum_1^\infty (-1)^i \epsilon_0^{i-1} \left(1 + \epsilon_0^2 + \frac{1 - \epsilon_0^2}{i}\right) \cos if, \quad e_0 = \frac{2\epsilon_0}{1 + \epsilon_0^2}.$$

7·13. The Fourier expansions which are needed above are obtained from

$$f - E_f = \int_0^f \frac{(1 - e^2)^{\frac{3}{2}}}{(1 + e \cos f)^2} df. \quad \text{......................(1)}$$

Differentiation with respect to e gives

$$\frac{\partial E_f}{\partial e} = \frac{1}{e} \int_0^f \frac{(1 - e^2)^{\frac{1}{2}}}{(1 + e \cos f)^2} \left\{ 2 + e^2 - 2 \frac{1 - e^2}{1 + e \cos f} \right\} df, \quad \text{.........(2)}$$

and differentiation of this result with respect to f provides

$$2\frac{(1-e^2)^{\frac{5}{2}}}{(1+e\cos f)^3}=(2+e^2)\frac{d}{df}(f-E_f)-e(1-e^2)\frac{\partial^2 E_f}{\partial e\partial f}. \quad \text{......(3)}$$

The value of $\partial E_f/\partial e$ can be obtained either from the series for E_f or by integrating (2). The latter integration is performed by making use of the identities,

$$\frac{\partial}{\partial f}\frac{\sin f}{1+e\cos f}=\frac{\cos f+e}{(1+e\cos f)^2},\quad \frac{\partial}{\partial f}\frac{\sin f}{(1+e\cos f)^2}=\frac{2e+\cos f-e\cos^2 f}{(1+e\cos f)^3},$$

the sum of the right-hand members being

$$\frac{3e+(2+e^2)\cos f}{(1+e\cos f)^3}=\frac{1}{e}\frac{2+e^2}{(1+e\cos f)^2}-\frac{2}{e}\frac{1-e^2}{(1+e\cos f)^3}.$$

Hence

$$\frac{\partial E_f}{\partial e}=(1-e^2)^{\frac{1}{2}}\left\{\frac{\sin f}{1+e\cos f}+\frac{\sin f}{(1+e\cos f)^2}\right\}$$

$$=\frac{(1-e^2)^{\frac{1}{2}}}{e}\frac{\partial}{\partial f}\left\{-\log(1+e\cos f)+\frac{1}{1+e\cos f}-1\right\}. \quad \text{......(4)}$$

The value of $F_f=D^{-1}(\partial E_f/\partial e)$ is obtained immediately from this result, and it also completes the expansion 7·13 (3).

The calculation of these expansions by harmonic analysis is at least as rapid as by series and is advisable for large values of e. Harmonic analysis can also be used to calculate E_f when $e=e_0+e_1\mathrm{v}$ by calculating the coefficients with $e=e_0$, and again with $e=e_0+ke_1$ when k has some convenient numerical value (? 100). The difference between the resulting pairs of coefficients divided by k gives the factor of v in the coefficient.

7·14. *The equation for* Γ. The equation 7·1 (7) for Γ gives, on integration,

$$\Gamma=(k+\Gamma_1\mathrm{v}+\Gamma_p)\,q^{\frac{1}{2}}, \quad \text{..................(1)}$$

where Γ_1 is the constant term in the expansion of $-q^{\frac{1}{2}}\partial R/u^2\partial\theta$, Γ_p is a sum of periodic terms, and k is an arbitrary constant.

The value 7·10 (2) for q gives, on expansion to the first order,

$$q^{\frac{1}{2}}=(1-e^2)^{-\frac{1}{2}}\{1+\tfrac{1}{2}(\delta q_0+q_p)(1-e_0^2)\}$$

$$=(1-e_0^2)^{-\frac{1}{2}}+e_1e_0(1-e_0^2)^{-\frac{3}{2}}\mathrm{v}+\tfrac{1}{2}(\delta q_0+q_p)(1-e_0^2)^{\frac{1}{2}},$$

whence

$$\Gamma=k(1-e_0^2)^{-\frac{1}{2}}+\tfrac{1}{2}k\delta q_0(1-e_0^2)^{\frac{1}{2}}+\{ke_1e_0(1-e_0^2)^{-\frac{3}{2}}$$

$$+\Gamma_1(1-e_0^2)^{-\frac{1}{2}}\}\mathrm{v}+\tfrac{1}{2}k(1-e_0^2)^{\frac{1}{2}}q_p+(1-e_0^2)^{-\frac{1}{2}}\Gamma_p.$$

The first two terms in this expression constitute the constant part of Γ; denote it by Γ_0. In terms which have the disturbing

mass as a factor we can put $k = \Gamma_0 (1 - e_0^2)^{\frac{1}{2}}$. Hence the value of Γ is given by

$$\Gamma = \Gamma_0 + \{e_1 e_0 \Gamma_0 (1 - e_0^2)^{-1} + \Gamma_1 (1 - e_0^2)^{-\frac{1}{2}}\} \mathrm{v} + \tfrac{1}{2}(1 - e_0^2)\,\Gamma_0 q_p + (1 - e_0^2)^{-\frac{1}{2}}\,\Gamma_p.$$

When the disturbing forces are neglected we have $\Gamma = \Gamma_0$, which is therefore the value of Γ used in calculating the perturbations. The secular motion of Γ is the coefficient of v. All the terms contain Γ_0 as a factor.

Since $\Gamma = 1 - \cos I$, $\delta\Gamma = \sin I\,\delta I$, the equation for Γ, which may be written $\Gamma = \Gamma_0 + \delta\Gamma$, gives $I = I_0 + \delta\Gamma/\sin I_0$, where $\delta\Gamma$ contains $\Gamma_0 = \sin^2 I_0/(1 + \cos I_0)$ as a factor. Hence when I_0 is small the perturbations of I have I_0 as a factor.

7·15. *The equation for θ.* The integral of 7·1 (8) gives

$$\theta = \theta_0 + \theta_1 \mathrm{v} + \theta_p,$$

where the signification of the symbols is evident. Here θ_0 is the value of θ used in calculating the perturbations and $\theta_1 \mathrm{v}$ is its 'secular motion.'

7·16. *The equation for v.* Since $d\theta/d\mathrm{v}$ contains the disturbing mass as a factor, we can put $\Gamma = \Gamma_0$ in 7·1 (6) so the integral is

$$v = \mathrm{v} - \Gamma_0 (\theta_1 \mathrm{v} + \theta_p),$$

no constant being added, in accordance with the definition in the last paragraph of 7·3.

The usual definition of n is to make it the mean value of dv/dt rather than of $d\mathrm{v}/dt$ as defined in 7·1 (6). This definition requires us to replace n by $n/(1 - \Gamma_0 \theta_1)$. Since n is not present in the coefficients of elliptic motion and since the change may be neglected in the perturbations, no further adjustment of the value of t in terms of v is necessary. The value of δq_0 (7·11) is altered and receives an additional part $\frac{2}{3}\Gamma_0 \theta_1/(1 + e_0^2)$; this is usually insensible in the value of u and elsewhere.

The only further change necessary to obtain t in terms of v is the replacement of v by $v + \Gamma_0 \theta_p$.

7·17. *The equation for sin L.* When $\Gamma\theta_p$ can be neglected, we can save some labour by integrating the equation for $\sin L$ to replace those for Γ, θ. If we treat the right-hand member like that of the equation for u, isolating the terms with argument f and also the constant term, the equation takes the form

$$(D^2+1)\sin L = g_0 + g_1\cos f + g_1'\sin f + g_p, \quad\text{..............(1)}$$

where g_p denotes the remaining periodic terms with arguments $jf+j_1 f_1$. The solution, like that for u, may be written

$$\sin L = \sin I_0 \sin(f+\varpi_0-\theta_0) + g_0 - \tfrac{1}{2}g_1'\,\mathrm{v}\cos f + \tfrac{1}{2}g_1\,\mathrm{v}\sin f + \frac{1}{D^2+1}g_p. \quad\text{......(2)}$$

The constants are shown in the above form because we have in ellipti motion

$$\sin L = \sin I_0 \sin(\mathrm{v}-\theta_0) = \sin I_0 \sin(f+\varpi_0-\theta_0).$$

The terms with factor v may be included in the solution

$$\sin L = \sin(I_0+I_1\mathrm{v})\sin(f+\varpi_0-\theta_0-\theta_1\mathrm{v}) + g_0 + (D^2+1)^{-1}g_p, \quad\text{...(3)}$$

if we put $I_1\mathrm{v}=\delta I$, $\theta_1\mathrm{v}=\delta\theta$, where δI, $\delta\theta$ are determined from

$$\delta\{\sin I\sin(\varpi_0-\theta)\} = -\tfrac{1}{2}g_1'\mathrm{v},\quad \delta\{\sin I\cos(\varpi_0-\theta)\} = \tfrac{1}{2}g_1\mathrm{v}. \quad\text{...(4)}$$

7·18. *The small divisors.* The divisors s are present in the equation for q, and when s is small, that is, when the corresponding term has a long period compared with $2\pi/n$, the coefficient will be increased by the integration. It is again increased by the same divisor in the integration of the equation for t, so that in the expression for t in terms of v or v, the divisor s^2 is present.

The divisor $1-s^2$ is small in the expression for u or $\sin L$ when s is nearly equal to ± 1, that is, when the period of the term is near that of revolution of the disturbed body. It might be expected that these terms in combination with the elliptic terms would produce terms with the product of small divisors s, $1-s^2$ in t. It is true that they do so, but such terms in general have a factor e^2 as compared with the terms in q from which they arise, a result which will be evident when the following method of the variation of the elements is used.

C. EQUATIONS FOR THE VARIATIONS OF THE ELEMENTS

7·19. Let us return to the original equations of motion in 7·1 and introduce three new variables a, e, ϖ to take the place of u, q. *These new variables have as yet no relation to those denoted by the same letters in the previous sections of this chapter.*

We retain the notation $D = d/d\mathrm{v}$ and, for brevity of expression, introduce new operators defined by

$$D_1 = Da\frac{\partial}{\partial a} + De\frac{\partial}{\partial e} + D\varpi\frac{\partial}{\partial \varpi}, \quad D_0 = \frac{\partial}{\partial \mathrm{v}}. \quad \text{......(1)}$$

Thus when we are operating on a function of a, e, ϖ, v we have $D = D_0 + D_1$.

Since we are replacing two variables by three, one relation between the new variables is at our disposal. The three relations to be adopted are

$$u = \frac{1 + e\cos(\mathrm{v} - \varpi)}{a(1 - e^2)}, \quad q = \frac{1}{a(1 - e^2)}, \quad D_1\frac{1 + e\cos(\mathrm{v} - \varpi)}{a(1 - e^2)} = 0, \quad \text{......(2)}$$

so that $$Du = D_0 u, \quad D_1 u = 0, \quad \text{..................(3)}$$

when u is expressed in terms of the new variables.

Since $Dq = D_1 q$, the third of equations (2) gives

$$\{1 + e\cos(\mathrm{v} - \varpi)\}\,Dq + qD_1\{e\cos(\mathrm{v} - \varpi)\} = 0.$$

This, combined with 7·1 (4), gives

$$D_1\{e\cos(\mathrm{v} - \varpi)\} = \frac{2}{u}\frac{\partial R}{\partial v}. \quad \text{...............(4)}$$

Next, the equation $Du = D_0 u$ gives

$$Du = -qe\sin(\mathrm{v} - \varpi). \quad \text{..................(5)}$$

Hence

$$\begin{aligned} D^2u + u - q &= -D\{qe\sin(\mathrm{v} - \varpi)\} + u - q \\ &= -D_1\{qe\sin(\mathrm{v} - \varpi)\} \\ &= -Dq\,.\,e\sin(\mathrm{v} - \varpi) - qD_1\{e\sin(\mathrm{v} - \varpi)\}, \end{aligned}$$

since $$D_0\{qe\sin(\mathrm{v} - \varpi)\} = u - q.$$

But $$-Dq\,.\,e\sin(\mathrm{v} - \varpi) = (Dq/q)\,.\,Du.$$

Whence, from 7·1 (3),

$$D_1\{e\sin(\mathrm{v}-\varpi)\} = -\frac{\partial R}{\partial u} - \frac{Du}{u^2}\frac{\partial R}{\partial v}. \qquad \text{.........(6)}$$

The equations (4), (6) are those which give the variations of e, ϖ.

To obtain the equation satisfied by a, multiply equations 7·1 (3), (4) by Du/q, $\{(Du)^2+u^2\}/2q^2$ respectively, and subtract; the result may be written

$$\tfrac{1}{2}D\left\{\frac{(Du)^2+u^2}{q} - 2u\right\} = \frac{\partial R}{\partial v} + Du\frac{\partial R}{\partial u}.$$

Substituting for u, Du, q from (2), (5), we obtain

$$D\left(-\frac{1}{2a}\right) = \frac{\partial R}{\partial v} + Du\frac{\partial R}{\partial u}, \qquad \text{..............(7)}$$

which is the equation satisfied by a.

7·20. The last equation may be transformed into a form which is not only more convenient for calculation but which furnishes an important theorem concerning the secular terms.

The disturbing function was originally expressed as a function of u, v, Γ, θ, and of t through its presence in r', v'. Hence

$$DR = \frac{\partial R}{\partial v}Dv + \frac{\partial R}{\partial u}Du + \frac{\partial R}{\partial \Gamma}D\Gamma + \frac{\partial R}{\partial \theta}D\theta + \frac{\partial R}{\partial t}Dt.$$

With the use of the expressions 7·1 (6), (7), (8) for Dv, $D\Gamma$, $D\theta$, this reduces to

$$DR = \frac{\partial R}{\partial v} + \frac{\partial R}{\partial u}Du + \frac{\partial R}{\partial t}Dt. \qquad \text{..............(1)}$$

Now t enters into r', v' only in the form $n't+\epsilon'$, and this angle enters (7·5) into R only through f_1 in such a manner that $\partial f_1/\partial t = n'$. Hence

$$\frac{\partial R}{\partial t} = n'\frac{\partial R}{\partial f_1}, \qquad \text{.......................(2)}$$

a result which is true whether R be expressed in terms of the old or new variables. Utilising (1), (2), we may write 7·19 (7) in the forms

$$D\left(\frac{1}{2a}+R\right) = n'\frac{\partial R}{\partial f_1}Dt = n'\left(\frac{q}{\mu}\right)^{\frac{1}{2}}\frac{\partial}{\partial f_1}\left(\frac{R}{u^2}\right), \qquad \text{......(3)}$$

since $Dt = q^{\frac{1}{2}}\mu^{-\frac{1}{2}}u^{-2}$ and since u does not contain f_1 explicitly.

7·21. Finally, the expression for Dt, just quoted, gives, in terms of the new variables with $n^2a^3 = \mu$ and the expansion 3·8 (3),

$$Dt = \frac{1}{n}\frac{(1-e^2)^{\frac{3}{2}}}{\{1+e\cos(\mathrm{v}-\varpi)\}^2} = \frac{1}{n} - \frac{1}{n}\frac{\partial E_f}{\partial \mathrm{v}}, \quad \text{......(1)}$$

where E_f is the equation of the centre. Hence

$$t = \int \frac{1}{n}\,d\mathrm{v} - \int D_0\left(\frac{E_f}{n}\right)d\mathrm{v} + \text{const.} \quad \text{............(2)}$$

The values of $n = (\mu/a^3)^{\frac{1}{2}}$, e, ϖ, as deduced from the integrals of 7·19 (4), (6), (7) are to be substituted under the integral signs in (2), and the integrations are then to be carried out.

Equation (2) may be written, since $D_0 = D - D_1$,

$$t = \int \frac{1}{n}\,d\mathrm{v} - \frac{E_f}{n} + \int D_1\left(\frac{E_f}{n}\right)d\mathrm{v} + \text{const.} \quad \text{.........(3)}$$

7·22. As in the earlier work, it is advisable to expand $Rr^2 = R_1$ rather than R. With this change and with the usual abbreviation, $f = \mathrm{v} - \varpi$, equations 7·19 (4), (6) can be written

$$De\,.\cos f + eD\varpi\,.\sin f = 2u\frac{\partial R_1}{\partial v}, \quad \text{............(1)}$$

$$De\,.\sin f - eD\varpi\,.\cos f = r^2\frac{\partial R}{\partial r} - Du\frac{\partial R_1}{\partial v}, \quad \text{......(2)}$$

the computation of $r^2\partial R/\partial r$ being carried out in the manner explained in 7·6. From these equations we deduce those for De, $eD\varpi$.

The equation for a becomes

$$D\left(\frac{1}{2a}\right) = \frac{n'}{a^{\frac{1}{2}}(1-e^2)^{\frac{1}{2}}}\frac{\partial R_1}{\partial f_1} - D(R_1u^2). \quad \text{.........(3)}$$

The equations for θ, Γ remain the same, namely

$$D\theta = \frac{1}{a(1-e^2)}\frac{\partial R_1}{\partial \Gamma}, \quad D(\Gamma q^{-\frac{1}{2}}) = -q^{\frac{1}{2}}\frac{\partial R_1}{\partial \theta} \text{....(4), (5)}$$

The latter may be replaced by

$$D\Gamma = -\frac{1}{a(1-e^2)}\left(\frac{\partial R_1}{\partial \theta} + \Gamma\frac{\partial R_1}{\partial v}\right). \quad \text{............(6)}$$

Equation 7·21 (3) is used to find t.

7·23. *Solution of the equations.* We proceed as before. When R is neglected a, e, ϖ, Γ, θ become constants, and the motion is elliptic. These constant values are substituted in the expansion of R_1, the derivatives of R_1 and of R being obtained as in 7·6. For the arbitrary constants we use these same values, except in the case of $1/a$, to the elliptic value of which we add α, where α is so determined that when the equation for Dt has been formed, the constant term shall be represented by $1/n_0$, where n_0 is the observed value of the mean motion.

The amount of calculation needed with the use of these equations is not very much greater than that required in the previous form. The additional work is mainly the multiplication of R_1 by the three terms in u^2, namely,

$$q^2 \{1 + \tfrac{1}{2} e^2 + 2e \cos f + \tfrac{1}{2} e^2 \cos 2f\},$$

where q^2, e are numerical constants, in order to find $1/a$, and the multiplications by $\cos f$, $\sin f$ to find e, ϖ.

In either case the work is less laborious than when t is used as the independent variable, chiefly because, in the latter case, the single terms $\cos f$, $\sin f$ have to be replaced by Fourier series containing a number of terms corresponding to the highest power of e we need to retain.

The chief saving of labour in the use of the methods of this chapter over those of Chap. IV is due to the avoidance of the formation of derivatives with respect to e. There is great advantage in using the numerical value of e from the outset and if this be done we cannot find the derivative with respect to e. Further, the development, to secure a given degree of accuracy, requires the presence of one power of e higher in the latter case than in the former.

7·24. The proofs of certain theorems, quoted earlier, follow easily from the equations of variations.

To show that there are no secular terms of the first order in $1/a$ and none of the form $\kappa \mathrm{v}^2$ in t, we use equations 7·20 (3). To the first order, $\partial R_1/\partial f_1$ has the same value whether R_1 be expressed in terms of u, v, t, Γ, θ, or in terms of f, f_1 and the constant elements. In the latter case, $\partial R_1/\partial f_1$ will have no term free from the angle f_1 and consequently no constant term. Under the same conditions, R_1 has no term proportional to v, so that $1/a$ has no term containing the factor v.

The secular terms in e, ϖ are $e_1 \mathrm{v}$, $e_1 \varpi$. These, substituted in E_f, produce terms of the forms $\mathrm{v} \cos if$, $\mathrm{v} \sin if$, but no term of the form $\kappa \mathrm{v}$ to the first order. Since there is no term factored by v in $1/n$, there will be none of the form $\kappa \mathrm{v}^2$ in t.

Since
$$u = \frac{1}{a(1-e^2)} + \frac{e \cos(f - \varpi)}{a(1-e^2)},$$
the substitution $e = e_0 + e_1 \mathrm{v}$ in the former of these terms will produce a secular term in u. This differs from the case of u expressed in terms of t, where the absolute term is $1/a$ which has no secular part. This fact exhibits the artificial nature of the statement that the major axis has no secular part. When referred to actual coordinates, the existance of a secular part depends on the coordinates used.

7·25. We next show that the only terms with the divisors s^2 arise through the variable a. This is proved by 7·22 (6). For the integrals giving e, ϖ contain the first power of s only, so that E_f/n gives rise to this class of terms only. Also $D_1(E_f/n)$, which depends on Da, De, $D\varpi$, has no divisors and its integral will give rise to terms of the same kind. The integral giving a gives rise to divisors s and that of $1/n$, through $n^2 a^3 = \mu$, to divisors s^2.

This latter result which was stated in 7·18 is of some assistance in the numerical developments of the methods in the previous section. It shows that the terms of long period in which s is small are needed in u to a lower degree of accuracy than terms with the same argument in q. If we calculate the coefficients of the long period terms of $\partial R/\partial u$ to a lower degree of accuracy than those of $\partial R/\partial v$, but *retain the same number of places of decimals*, the theorem shows that the inaccurate or omitted portions cancel one another. The relative degree of accuracy needed must be judged from the small values of s present.

D. The Second Approximations to the Perturbations

7·26. The results of the first approximations are as follows. We have obtained u, q, t, v, Γ, θ in the forms
$$u = u_0 + u_p, \quad q = q_0 + q_p, \quad t = t_0 + t_p,$$
$$\Gamma = \Gamma_0 + \Gamma_1 \mathrm{v} + \Gamma_p, \quad \theta = \theta_0 + \theta_1 \mathrm{v} + \theta_p, \quad v = \mathrm{v} - \Gamma(\theta - \theta_0),$$
where
$$u_0 = \frac{1 + e\cos(f - \varpi_1 \mathrm{v})}{a_0(1-e^2)}, \quad q_0 = \frac{1}{a_0(1-e^2)},$$
$$n_0 t + \epsilon_0 = \mathrm{v} - E_f, \quad f = \mathrm{v} - \varpi_0,$$

in which $e = e_0 + e_1 \mathrm{v}$; E_f is the equation of the centre with eccentricity e, true anomaly $f - \varpi_1 \mathrm{v}$, and $a_0^3 n_0^2 = \mu$. The constants a_0, e_0, ϖ_0, Γ_0, θ_0 have been used to compute the terms arising from R; e_1, ϖ_1, θ_1, Γ_1 are constants whose values have been found; u_p, q_p, t_p, Γ_p, θ_p are sums of periodic terms, u_p, q_p alone containing a constant term, so that δq_0 of 7·10 (2) is now included in u_p, q_p.

If we had solved by the method of the variation of arbitrary constants, the forms of t, Γ, θ, v would have remained the same, but for u, q we should have had

$$a = a_0 + a_p, \quad e = e_0 + e_1 \mathrm{v} + e_p, \quad \varpi = \varpi_0 + \varpi_1 \mathrm{v} + \varpi_p,$$

to be substituted in

$$u = \{1 + e \cos(\mathrm{v} - \varpi)\} \div a(1 - e^2), \quad 1/q = a(1 - e^2).$$

7·27. To obtain the second approximation, these values of the variables, or of the elements, must be substituted in R in the place of the constant elements previously used. Whichever plan has been adopted in the first approximation, we can and shall still use the equations for the variations of the elements in the second approximation on account of their greater simplicity for both computation and exposition. The exposition will be limited by neglecting the variations of Γ, θ, $v - \mathrm{v}$. The effects of these variations on the perturbations of the second order are usually insensible, but they can be included, if necessary, by the use of the methods given for the other variables.

7·28. Denote any perturbation of the first order by the symbol δ and one of the second order by δ_2 and put $\mu = 1$.

Equation 7·20 (3) gives

$$\tfrac{1}{2} D\left(\delta_2 \frac{1}{a}\right) = n' \frac{\partial R_1}{\partial f_1} \delta q^{\frac{1}{2}} + n' q^{\frac{1}{2}} \delta \frac{\partial R_1}{\partial f_1} - D(\delta R), \quad \text{......(1)}$$

and

$$\delta \frac{\partial R_1}{\partial f_1} = \frac{\partial^2 R_1}{\partial f_1 \partial u} \delta u + \frac{\partial^2 R_1}{\partial f_1 \partial t} \delta t = \frac{\partial}{\partial f_1}\left(\frac{\partial R_1}{\partial u}\right) \delta u + n' \frac{\partial^2 R_1}{\partial f_1^2} \delta t, \quad \text{......(2)}$$

$$\delta R = \frac{\partial R}{\partial u} \delta u + n' \frac{\partial R}{\partial f_1} \delta t. \quad \text{..............................(3)}$$

Since R_1, $\partial R_1/\partial u$, $\partial R_1/\partial f_1$, R have already been obtained in finding the first approximation and are expressed in terms of the angles f, f_1, their derivatives with respect to f_1 are immediately obtained. If the first approximation has been obtained by finding the variations of the elements, we replace $\delta q^{\frac{1}{2}}$, δu by

$$\delta\{a^{-\frac{1}{2}}(1-e^2)^{-\frac{1}{2}}\},\quad \frac{\partial u_0}{\partial a_0}\delta a + \frac{\partial u_0}{\partial e_0}\delta e + \frac{\partial u_0}{\partial \varpi_0}\delta\varpi, \ldots\ldots(4)$$

where $$u_0 = \{1 + e_0 \cos(\mathrm{v} - \varpi_0)\} \div a_0(1 - e_0^2).$$

It will presently appear that we can usually neglect δa, δe, $\delta\varpi$ and therefore δu, $\delta D_0 u$, δq in finding $\delta_2 e$, $\delta_2\varpi$. When this is the case, equations 7·22 (1), (2) give

$$\cos f D(\delta_2 e) + e_0 \sin f D(\delta_2\varpi) = 2n' u \frac{\partial^2 R_1}{\partial f_1 \partial v}\delta t, \ldots\ldots\ldots\ldots\ldots(5)$$

$$\sin f D(\delta_2 e) - e_0 \cos f D(\delta_2\varpi) = n'\left\{\frac{\partial}{\partial f_1}\left(r^2\frac{\partial R}{\partial r}\right) - D_0 u \frac{\partial^2 R_1}{\partial f_1 \partial v}\right\}\delta t. \ldots\ldots(6)$$

The second order derivatives, again being derivatives with respect to f_1 of first order derivatives used in the first approximation, are obtained immediately. Further, only a very few terms in δu, δt have to be considered and the same is true of their products by the second derivatives of R or R_1. Finally, as these variations enter into the right-hand members of the equations in a linear form only, we can compute separately the effects due to the few terms in δu, δt which have to be taken into consideration.

7·29. *Calculation of the effects due to the secular terms in the first approximation.* To obtain them we put

$$\delta e = e_1 \mathrm{v},\quad \delta\varpi = \varpi_1 \mathrm{v},\quad \delta\left(\frac{1}{a}\right) = 0.$$

From these we get

$$\delta u = \frac{\partial u_0}{\partial e_0} e_1 \mathrm{v} + \frac{\partial u_0}{\partial \varpi_0}\varpi_1 \mathrm{v}.$$

These results are to be substituted in the right-hand members of 7·28 (1), (2), (3), (5), (6). They give terms of the form

$$\kappa \mathrm{v} + \mathrm{v}\,\Sigma C \sin(s\mathrm{v} + \alpha).$$

First, let us consider the equation 7·28 (1). The only portion of this equation which can give a term of the form κv from δu, is

$$-D\left\{\frac{\partial}{\partial u}(R_1u^2)\,.\,\delta u\right\}=-(D_0+D_1)\left\{\frac{\partial}{\partial u}(R_1u^2)\,.\,\delta u\right\},$$

since all the other portions are products of series one member of which contains f_1 and the other is independent of f_1. For terms of this form we therefore have, since $R=R_1u^2$,

$$\delta_2\left(\frac{1}{a}\right)=-2\frac{\partial}{\partial u}(R_1u^2)\left\{\frac{\partial u_0}{\partial e_0}e_1\mathrm{v}+\frac{\partial u_0}{\partial \varpi_0}\varpi_1\mathrm{v}\right\},$$

from which the terms of the form κv are to be chosen. This is equivalent to putting $e=e_0+e_1\mathrm{v}$, $\varpi=\varpi_0+\varpi_1\mathrm{v}$ in the constant term of the expansion of $-2R_1u^2$ or in $-2R$.

The expression 7·28 (2) may be written

$$\frac{\partial}{\partial f_1}\left(\frac{\partial R_1}{\partial u}\delta u+n'\frac{\partial R_1}{\partial f_1}\delta t\right)-\frac{\partial R_1}{\partial u}\frac{\partial}{\partial f_1}\delta u-\frac{\partial R_1}{\partial t}\frac{\partial}{\partial f_1}\delta t.$$

The two latter terms are equal to $-\partial(\delta R)/\partial f_1$. Hence when we substitute for δu, δt the portions u_p, t_p, 7·28 (3) will have no constant term and therefore will produce no term of the form κv in $\delta_2 a$. Evidently, these portions substituted in $D(R_1u^2)$ produce no such term. Hence, the only term of the form κv present in $\frac{1}{2}\delta_2(1/a)$ arises from the constant term in

$$\tfrac{1}{2}n'\frac{q_p}{q_0^{\frac{1}{2}}}\frac{\partial R_1}{\partial f_1},$$

which in general will not be zero. Thus the theorem, that $1/a$ has no secular term of the second order when t is the independent variable, is not true when v is the independent variable. It may be noted however that it is true for a second order perturbation arising from two different disturbing planets.

The same arguments evidently apply to the right-hand members of 7·28 (5), (6) which thus contain secular terms of the forms v cos $(jf+j_1f_1)$, v sin $(jf+j_1f_1)$, with $j_1 \neq 0$. Hence these portions of $\delta_2 e$, $\delta_2\varpi$ give no terms of the form $\kappa\mathrm{v}^2$ in the coordinates. Such terms, however, will arise from δu; when it is necessary to calculate them, we shall need those parts of the

second derivatives of R_1 with respect to u, v which are independent of f_1: their calculation presents little difficulty since the degree of accuracy required is quite low.

The practical importance of the theorem that there are no secular terms of the first and second orders with respect to the masses in $1/a$ when t is the independent variable, has been much over-estimated. It is a result which eliminates certain secular terms from $1/r$, but does not do so from r or, in general, from other functions of r. There is no particular reason from a physical point of view why, in getting a mean value of the deviation of the orbit from circularity, we should choose $1/r$ rather than r as the function to be averaged. The fact is that the separation of the deviations of the co-ordinates into deviations of a, e, ϖ, etc. is an artificial one, convenient for calculation and description, but one which has no particular physical significance.

7·30. Terms of the form $\mathrm{v}\cos s\mathrm{v}$, $\mathrm{v}\sin s\mathrm{v}$ are integrated by the formulae

$$\int \mathrm{v}\cos s\mathrm{v}\, d\mathrm{v} = \frac{\mathrm{v}}{s}\sin s\mathrm{v} + \frac{1}{s^2}\cos s\mathrm{v}, \quad \int \mathrm{v}\sin s\mathrm{v}\, d\mathrm{v} = -\frac{\mathrm{v}}{s}\cos s\mathrm{v} + \frac{1}{s^2}\sin s\mathrm{v}.$$

Such terms are present in $\delta_2 a$, $\delta_2 e$, $\delta_2 \varpi$ and therefore in $\delta_2 u$, $D\delta_2 t$. The succeeding integration necessary to obtain $\delta_2 t$ will introduce the factors v/s^2, $1/s^3$.

These terms will usually be insensible except when s is very small and even then the only portions which need be retained are those in δt having the last-named factors.

7·31. *Calculation of the effects due to the periodic terms in the first approximation.* Only the long period terms need to be considered. The right-hand members of the equations in 7·28 have the form—a derivative of R multiplied by δu or by δt. In δu, the divisor s is present; in δt, the divisors s, s^2 are present. The products just referred to will produce products

$$\cos s\mathrm{v} \,.\, \cos s'\mathrm{v} = \tfrac{1}{2}\cos(s - s')\,\mathrm{v} + \tfrac{1}{2}\cos(s + s')\,\mathrm{v},$$

with similar results for sines of $s\mathrm{v}$, $s'\mathrm{v}$. Long period terms can arise in two ways: from short period terms in which $s \pm s'$ is small, or from long period terms in which both s, s' are small.

The former will produce small divisors $(s \pm s')$ in $\delta_2 u$ and their squares in $\delta_2 t$. These will rarely be sensible.

For the latter, we have divisors s, s^2 or s', s'^2 in the right-hand members of the equations of 7·28, and therefore the smallest divisors present in their integrals will have the forms $s^2(s \pm s')$. Hence, the small divisors in $\delta_2 t$, arising from $\delta_2 a$, will have the form $s^2(s \pm s')^2$. Thus, whenever we are able to neglect the squares of the small divisors in the second approximation, the equations of 7·28 will be sufficient for the calculation of the remaining terms. Even then it is usually necessary to consider only one or two terms, so that the amount of calculation needed is quite limited.

We shall now show that the chief part of a long period term in $\delta_2 t$—that having the divisor $s^2(s \pm s')^2$—can be obtained immediately from the first approximation.

7·32. *Calculation of the portion of a coefficient in $\delta_2 t$ depending on the fourth power of the small divisor**.

We have seen in the previous paragraph that this portion can arise only from the term

$$\tfrac{1}{2} D\delta_2\left(\frac{1}{a}\right) = n'^2\left(\frac{q_0}{\mu}\right)^{\frac{1}{2}} \frac{\partial^2 R_1}{\partial f_1^2}\delta t, \quad \text{..............(1)}$$

substituted in

$$\delta_2 t = \int \delta_2\left(\frac{1}{n}\right) dv, \quad \text{.....................(2)}$$

where δt arises from

$$\tfrac{1}{2} D\delta\left(\frac{1}{a}\right) = n'\left(\frac{q_0}{\mu}\right)^{\frac{1}{2}} \frac{\partial R_1}{\partial f_1}, \quad \text{..............(3)}$$

substituted in

$$\delta t = \int \delta\left(\frac{1}{n}\right) dv, \quad \text{.....................(4)}$$

by the use of the relation $n^2 a^3 = \mu$.

Suppose that the term in R_1 is

$$a_0\beta \cos s\text{v} + a_0\beta' \sin s\text{v} = a_0 A \cos(s\text{v} + s_1), \qquad s\text{v} = jf + j_1 f_1. \quad \text{......(5)}$$

* E. W. Brown, *Mon. Not. R.A.S.* vol. 90, p. 14.

If this be substituted in (3) and the result integrated, we obtain

$$\delta\left(\frac{1}{a}\right) = 2n'\left(\frac{q_0}{\mu}\right)^{\frac{1}{2}}\frac{j_1}{s}a_0 A\cos(sv+s_1). \quad \text{.........(6)}$$

The equation $n^2a^3 = \mu$ gives

$$\delta\frac{1}{n} = -\frac{3}{2}\frac{a_0}{n_0}\delta\frac{1}{a}. \quad \text{....................(7)}$$

Whence, from (4),

$$\delta t = -3\frac{n'}{n_0}\left(\frac{q_0}{\mu}\right)^{\frac{1}{2}}a_0^2\frac{j_1}{s^2}A\sin(sv+s_1) = \frac{B}{n_0}\sin(sv+s_1), \quad \text{.........(8)}$$

where

$$j_1 n'\left(\frac{q_0}{\mu}\right)^{\frac{1}{2}}A = -\frac{s^2}{3}\frac{B}{a_0^2}. \quad \text{.................(9)}$$

From the substitution of (5) in (1), we have

$$\begin{aligned} D\delta_2\left(\frac{1}{a}\right) &= -2n'^2\left(\frac{q_0}{\mu}\right)^{\frac{1}{2}}j_1^2 A a_0\cos(sv+s_1)\,.\,\delta t \\ &= \frac{2}{3}\frac{n'j_1s^2}{a_0}B\cos(sv+s_1)\,.\,\delta t \\ &= \frac{1}{3}\frac{n'j_1s^2}{a_0n_0}B^2\sin 2(sv+s_1), \quad \text{...............(10)} \end{aligned}$$

from (8), (9). The integral gives

$$\delta_2\frac{1}{a} = -\frac{1}{6}\frac{n'j_1s}{a_0n_0}B^2\cos 2(sv+s_1). \quad \text{.........(11)}$$

And since $\{\delta(1/a)\}^2$ has the divisor s^2 only, while $\delta_2(1/a)$ has the divisor s^3, we have, neglecting the former,

$$\delta_2\left(\frac{1}{n}\right) = -\frac{3}{2}\frac{a_0}{n_0}\delta_2\left(\frac{1}{a}\right).$$

Finally, $$\delta_2 t = \int\delta_2\left(\frac{1}{n}\right)dv = -\frac{3}{2}\int\frac{a_0}{n_0}\delta_2\left(\frac{1}{a}\right)dv,$$

which, with the use of (11), gives

$$\delta_2 t = \frac{1}{8}\frac{n'}{n_0^2}j_1B^2\sin 2(sv+s_1). \quad \text{...........(12)}$$

7·33. If we have two long period terms in R_1, namely,

$$R_1 = a_0 A \cos(s\mathrm{v} + s_1) + a_0 A' \cos(s'\mathrm{v} + s_1'),$$

and if we wish to obtain the terms due to their combinations, it is evident from equation 7·32 (10) that we shall have

$$D\delta_2\left(\frac{1}{a}\right) = \frac{2}{3}\frac{n' j_1 s^2}{a_0 n_0} B\cos(s\mathrm{v} + s_1) . B'\sin(s'\mathrm{v} + s_1')$$

$$+ \frac{2}{3}\frac{n' j_1' s'^2}{a_0 n_0} B'\cos(s'\mathrm{v} + s_1') . B\sin(s\mathrm{v} + s_1)$$

$$= \frac{1}{3}\frac{n'}{a_0 n_0} BB' (s^2 j_1 \pm s'^2 j_1') \sin\{(s' \pm s)\mathrm{v} + s_1' + s_1\},$$

the upper sign giving one term and the lower the other.

The process previously followed gives

$$\delta_2 t = \frac{1}{2}\frac{n'}{n_0^2} BB' \frac{s^2 j_1 \pm s'^2 j_1'}{(s' \pm s)^2} \sin\{(s' \pm s)\mathrm{v} + s_1' \pm s_1\}.$$

In the case $s' = s$, $s_1' \mp s_1$, the term arising from the lower signs disappears since $j_1 = j_1'$. The remaining term gives

$$\delta_2 t = \frac{1}{8}\frac{n'}{n_0^2} j_1 BB' \sin(2s\mathrm{v} + s_1 + s_1').$$

7·34. The same method may be applied to find the effect of a perturbation $\delta_1 v'$ of the longitude of the disturbing planet. In 7·5 we replace v' by $f_1 + \varpi' +$ equation of the centre, so that $\delta_1 v'$ may be regarded as an addition to f_1. Hence in this case the addition to $\partial R/\partial f_1$ is

$$\delta_1\left(\frac{\partial R_1}{\partial f_1}\right) = \frac{\partial^2 R_1}{\partial f_1^2}\delta_1 v'.$$

If $\delta_1 v' = B_1' \sin(s'\mathrm{v} + s_1')$ we can therefore utilise the formulae 7·32 (11), (12) by putting B_1'/n' for B'/n. The latter formula, in particular, gives an additive part to t:

$$\delta_2 t = \frac{1}{8}\frac{j_1}{n_0} BB_1' \sin(2s\mathrm{v} + s_1 + s_1').$$

7·35. *Numerical illustration. The 'great inequality' of Jupiter and Saturn.* The periods of revolution of Jupiter and Saturn are very nearly in the ratio of 2 to 5, so that if n, n' be their mean motions, the terms with argument $(5n' - 2n)t$ will have a very long period—actually about 70 times

that of Jupiter. Thus, in the motion of Jupiter disturbed by Saturn, $1/s^4$ will have the order $2{\cdot}5 \times 10^7$.

Suppose that we have calculated the terms with argument $(5n' - 2n)\,t$ to the first order in the motions of both planets, and that we need the principal second order portions, the latter can be obtained immediately from the formulae in 7·32. We shall perform the calculation and compare the results with those given by Hill*. Since the latter uses Hansen's method it will be necessary to compare it with that of this chapter.

Denote Hill's notation by (H) and that of this chapter by (v). In elliptic motion we have

$$(H),\ v = z + \varpi + E(z);\quad (\mathrm{v}),\ nt + \epsilon = v - E(f),$$

where $E(z)$ is the equation of the centre expressed in terms of the mean anomaly z and $E(f)$ is the same expressed in terms of the true anomaly f. In disturbed motion we have

$$(H),\ v = z + \delta z + \varpi + E(z + \delta z);\quad (\mathrm{v}),\ nt + \epsilon = v - E(f) + n\,\delta t.$$

Now we have seen that, for the principal part of a long period perturbation of the first order, the portion due to the elliptic periodic terms can be neglected since it only produces portions with the divisor s. Hence, very nearly,

$$n\,\delta t = -n\,\delta z,\quad n'\,\delta t' = -n'\,\delta z',$$

the former for Jupiter and the latter for Saturn.

There are two resulting second order perturbations in the motion of each planet. The first is that which arises as in 7·32 and the second that which arises from substituting the disturbed motion of the disturbing planet in R. For the latter we have in the motions of Jupiter and Saturn, respectively,

$$\delta v' = n'\,\delta z',\quad \delta v = n\,\delta z,$$

since there are additions to v', v in the respective disturbing functions.

Hill gives†

$$n\,\delta z = 1196''\sin N,\quad n'\,\delta z' = -2908''\sin N,$$

where $N = 5g' - 2g + 69°$, approximately. (There are additional secular parts given by Hill arising from the second order terms.) Thus for Jupiter the two portions are, in our notation,

$$n\,\delta t = -1196''\sin N,\quad \delta v' = -2908''\sin N,$$

and in N we can put f for $nt + \epsilon - \varpi = g$ and f_1 for $n't + \epsilon' - \varpi' = g'$, for the reasons stated above.

If $\rho = 1/206265''$, the factor necessary to reduce the coefficient to radians, we have, in the notation of 7·32, $B = -1196\rho$, $B_1 = -2908\rho$, $j_1' = 5$, $n'/n_0 = 2/5$ and the two parts give

$$n_0\delta_2 t = \tfrac{5}{8}\,.\,1196\rho\,(\tfrac{2}{5}\,.\,1196\rho + 2908\rho)\sin 2N = 12''{\cdot}3\sin 2N.$$

* *American Ephemeris Papers*, vol. 4; *Coll. Works*, vol. 3.

† *Coll. Works*, vol. 3, pp. 560, 568.

For the action of Jupiter on Saturn we have similarly,

$$n_0\delta_2 t = -\tfrac{2}{8}\,.\,2908\rho\,(\tfrac{5}{2}\,.\,2908\rho+1196\rho)\sin 2N = -30''\!\cdot\!9\sin 2N.$$

Hill's coefficients for this term in $n\,\delta z$, $n'\,\delta z'$ are*

Jupiter (first + second approx.) – (first approx.) = $-11''\!\cdot\!0-1''\!\cdot\!4=-12''\!\cdot\!4$,
Saturn „ „ „ „ „ = $26''\!\cdot\!8+3''\!\cdot\!4=\ 30''\!\cdot\!2$.

After the change of sign necessary for accordance with our notation, the comparison shows a close agreement between the two sets of results.

It may be pointed out that the percentage accuracy of the method is greater the smaller the value of s. As far as the present demands for accuracy in the various problems of the planetary theory are concerned, the method will give all second order coefficients with sufficient accuracy. It has been shown, however, that the terms depending on the divisor s^3 can be obtained without great labour.

E. TRANSFORMATION TO THE TIME AS INDEPENDENT VARIABLE

7·36. Although most of the purposes for which the theory is developed can be equally well served whether v or t be used as the independent variable, comparisons of different theories will be facilitated if we can transform from one to the other without too much labour.

The method gives results in the form

$$n_0 t+\epsilon = \mathrm{v}-E_f+P, \quad v=\mathrm{v}+\delta v, \quad \ldots\ldots(1),\ (2)$$

where E_f is the equation of the centre expressed in terms of the true anomaly f, and P, δv are the perturbations expressed in terms of v by meaus of

$$f=\mathrm{v}-\varpi, \quad f_1=\frac{n'}{n}(\mathrm{v}-\epsilon)+\epsilon'-\varpi'. \quad \ldots\ldots(3),\ (4)$$

The expression of v in terms of t can be deduced from (1) by the use of Lagrange's theorem, but the following method of carrying out the transformation in two steps demands much less calculation.

Put $n_0 t+\epsilon-\varpi=g$ and write (1) in the form

$$g-P=f-E_f. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

* *Coll. Works*, vol. 3, pp. 195, 107, 561, 569.

This equation may be regarded as expressing a 'mean anomaly' $g-P$ in terms of the true anomaly f. The formula 3·11 (6) which gives the true anomaly in terms of the mean may therefore be used. It gives

$$\begin{aligned} f &= g-P+E_{g-P} \\ &= g-P+2e\sin(g-P)+\tfrac{5}{4}e^2\sin 2(g-P)+\ldots \\ &= g-P+E_g-P\frac{dE_g}{dg}+\tfrac{1}{2}P^2\frac{d^2E_g}{dg^2}+\ldots, \qquad\ldots\ldots\ldots\ldots(6) \end{aligned}$$

by Taylor's theorem. The terms dependent on P^2 will be very small and higher powers of P may be neglected.

The perturbations P are expressed in terms of f, f_1, and their expression in terms of t by continued approximation constitutes the second step. The first approximation consists in putting in them

$$f=g+E_g, \quad f_1=g'+\frac{n'}{n}E_g, \qquad\ldots\ldots\ldots\ldots\ldots(7)$$

where $\qquad g'=n't+\epsilon'-\varpi'.$

Consider any term of P:

$$B\sin(jf+j_1f_1+B_1),$$

where B, B_1 are constants. With the use of (7) this term becomes

$$B\sin(jg+j_1g'+B_1)\cos\left\{\left(j+j_1\frac{n'}{n}\right)E_g\right\}$$
$$+B\cos(jg+j_1g'+B_1)\sin\left\{\left(j+j_1\frac{n'}{n}\right)E_g\right\}.$$

The second factors of these are expressible as Fourier series with argument g, either by harmonic analysis or by the respective formulae

$$1-\tfrac{1}{2}\left(j+j_1\frac{n'}{n}\right)^2E_g^2+\ldots, \quad \left(j+j_1\frac{n'}{n}\right)E_g-\ldots, \quad\ldots(8)$$

for expansions of sines and cosines in terms of the angles. An important point to notice is the fact that for long period terms $j+j_1\,.\,n'/n$ is small, so that the effect of the transformation in changing the terms which usually have the largest coefficients is small.

The transformed value of P is substituted in (6), which then gives f in terms of t to the first order of the disturbing forces.

If this first approximation be denoted by $f = g + E_g + \delta f$, a second approximation is obtained by replacing E_g by $E_g + \delta f$ in (8), and adding the second order term in (6) with $\delta f = 0$.

For the great majority of the terms in P, powers of the eccentricity beyond the first in the transformation may be neglected. For all such cases, a perturbation

$$B \sin (jf + j_1 f_1 + B_1)$$

in $n_0 t$ becomes a perturbation

$$-B \sin (jg + j_1 g' + B_1) . (1 + 2e \cos g) - B \cos (jg + j_1 g' + B_1) . 2e \sin g$$

in v, where $s = j + j_1 . n'/n$.

For the final step we have

$$v - \varpi = f + v(f, f_1),$$

where the last term contains the perturbations due to the transformation from an origin in the osculating plane to one in the fixed plane. These, having the square of the inclination as well as the disturbing mass as factors, are very small, so that the values (5), (6) for f, f_1 with $E_g = 2e \sin g$ will serve.

The remaining coordinates $1/r$, θ, Γ are transformed in a similar manner, that of $1/r$ being found with the aid of 2·2 (2). The value just found for f in terms of t is used in the expression for $1/r$, while the values (5), (6) will be sufficiently accurate for substitution in the expression for θ, Γ.

7·37. The method of this chapter is closely allied, as far as its final form is concerned, with that of Hansen*. Substantially, his method requires the expression of the true longitude in the form

$$v = g + \varpi + \delta g + 2e \sin (g + \delta g) + \tfrac{5}{4} e^2 \sin 2 (g + \delta g) + \dots,$$

so that all the perturbations are expressed by adding δg to the mean anomaly g. Equations 7·36 (1), (2) show that the same thing is done here, with the difference, however, that while Hansen calculated δg in terms of t, it is here calculated in terms of v.

That the theory is more simple than that of Hansen is due to the fact that it can be expressed by means of equations which follow forms well

* The original theory is given in a volume, *Fundamenta Nova, etc.*, Gotha, 1838.

known in other dynamical problems. The two principal objections which may be urged are, first, the necessity for expressing the true longitude of the disturbing planet as a function of that of the disturbed planet, and second, the possible need for the final transformation given in 7·36. As we have seen, the latter requires comparatively little additional calculation, while it is doubtful whether the former transformation, which is needed only in the development of the disturbing forces, requires more labour than that in terms of t. As pointed out in Chap. IV, each requires substantially four operations, two of which are more simple in the present method than in that of Hansen, while one of the remaining operations is more complicated.

There is, however, a special advantage possessed by the present method when we are dealing with the perturbations produced by an exterior planet on an interior one having a considerable eccentricity. In the transformation giving v' as a function of f, the principal elliptic term, $2e \sin f$, enters only with the factor n'/n which in most cases is not much greater than 1/2. Thus the powers of this elliptic term have a maximum effect nearer to those of e than to those of $2e$. To some extent this is compensated by the factor r^2 which accompanies R in the developments, but the convergence is much more easily controlled with the method of this chapter than with the usual independent variable.

F. Approximate formulae for the Perturbations

7·38. It is often useful to get an idea of the order of magnitude of the perturbations in a given problem. This is particularly the case when extensive calculations are to be undertaken to obtain the general perturbations accurately; an approximate preliminary calculation may save much unnecessary labour. Approximate formulae can also be utilised when the interval of time during which the results are needed is short or when the constants of the orbit are not well known.

In obtaining such formulae, we shall neglect the inclination, so that $v = \mathrm{v}$, and attention can be confined to the equations 7·1 (3), (4), (5). If in these equations we put

$$\left.\begin{aligned} 1/q_0 = a_0(1-e^2), \quad q = (1+\delta q)\,q_0, \quad u = (1+e\cos f+\delta u)\,q_0, \\ a_0^3 n_0^2 = \mu, \quad n_0 t + \epsilon = (1-e^2)^{\frac{3}{2}}(1+e\cos f)^{-2} + n_0\,\delta t, \end{aligned}\right\} \quad \ldots\ldots(1)$$

hey may be written

$$\frac{d}{dv}\,\delta q = -2\frac{\partial}{\partial v}(q_0 r^2 R), \quad\text{.......................(2)}$$

$$\left(\frac{d^2}{dv^2}+1\right)\delta u = \delta q - r^2\frac{\partial R}{\partial r} - \tfrac{1}{2}e\sin f\frac{d}{dv}\,\delta q, \quad\text{..............(3)}$$

$$n_0\frac{d}{dv}\,\delta t = \frac{(1-e^2)^{\frac{3}{2}}}{(1+e\cos f)^2}\left(\tfrac{1}{2}\delta q - 2\frac{\delta u}{1+e\cos f}\right)$$

$$= \tfrac{1}{2}\delta q - 2\delta u + (6\delta u - \delta q)\,e\cos f + \ldots. \quad\text{...(4)}$$

Consider a term with argument σ in the disturbing function and suppose that this term gives rise to terms

$$A\sin\sigma \;\text{ in }\; -\frac{\partial}{\partial v}(r^2 R q_0), \qquad B\cos\sigma \;\text{ in }\; r^2\frac{\partial R}{\partial r}. \quad\text{......(5), (6)}$$

In general, A, B will be of the same order of magnitude.

Substituting (5) in (2) and integrating, we obtain

$$\delta q = -\frac{2A}{s}\cos\sigma, \qquad s = \frac{d\sigma}{dv}. \quad\text{..............(7)}$$

With the help of (5), (6), (7), equation (3) becomes

$$\left(\frac{d^2}{dv^2}+1\right)\delta u = -\left(\frac{2A}{s}+B\right)\cos\sigma + \tfrac{1}{2}eA\cos(\sigma+f) - \tfrac{1}{2}eA\cos(\sigma-f),$$

the integral of which furnishes

$$\delta u = -\left(\frac{2A}{s}+B\right)\frac{\cos\sigma}{1-s^2} - \frac{eA}{2s}\left\{\frac{\cos(\sigma+f)}{2+s} + \frac{\cos(\sigma-f)}{2-s}\right\}, \quad\text{......(8)}$$

since $1-(s\pm1)^2 = \mp 2s - s^2$.

The substitution of (7), (8) in (4) and a subsequent integration give $n_0\delta t$.

7·39. Let us first neglect the terms in 7·38 (8), (4) which have the explicit factor e. The remaining terms give

$$\delta q = -\frac{2A}{s}\cos\sigma, \qquad \delta u = -\left(\frac{2A}{s}+B\right)\frac{\cos\sigma}{1-s^2}, \quad\text{...(1), (2)}$$

$$n_0\delta t = \left\{-\frac{A}{s^2} + \frac{2}{1-s^2}\left(\frac{2A}{s^2}+\frac{B}{s}\right)\right\}\sin\sigma. \quad\text{.........(3)}$$

The transformation to the time as independent variable is immediately made, since with $e=0$, we have $\delta v = -n_0 \delta t$, and $f=g, f_1=g'$ in σ.

These results hold for all terms in the disturbing function whether they contain e or not; we have neglected e only where it appears explicitly in 7·38 (3), (4) and in the transformation to the time as independent variable. Denote the former by the suffix zero, and the additional parts factored by the first power of e by the suffix unity. To obtain the results to the first power of e, it is sufficient to substitute the terms with suffix zero in the previously neglected terms. Hence

$$\delta_1 u = -\frac{e}{2}\left\{\frac{A}{s}\,\frac{\cos(\sigma+f)}{2+s} + \frac{A}{s}\,\frac{\cos(\sigma-f)}{2-s}\right\}_0, \quad \ldots(4)$$

$$n_0 \frac{d}{dv}\delta_1 t = -2\delta_1 u + (6\delta_0 u - \delta_0 q)\, e\cos f. \quad \ldots\ldots\ldots\ldots\ldots(5)$$

To the same order, the formulae for the transformation to the time as independent variable give

$$\delta v = -n_0 \delta t\,(1+2e\cos g) - 2e\sin g\,\frac{d}{dt} n_0 \delta_0 t - n_0 \delta_1 t, \quad \ldots\ldots(6)$$

with $f=g, f_1=g'$ in all the formulae.

The terms of chief importance are usually those in which s or $s-1$ is small and the order of magnitude of these is given by $\delta_0 v = -n_0 \delta_0 t$. When $s-2$ is small the additional terms with argument $\sigma - f$ must be considered.

7·40. *Solution to the first powers of the eccentricities.* The development of $a'/\Delta - a'r\cos S/r'^2$ as far as this order is*

$$\Sigma_i \{a_i \cos i\psi - e(D-2\mathrm{m}i)\, a_i \cos(i\psi + f) + e'(D+1-2i)\, a_i \cos(i\psi + f_1)\}$$
$$+\alpha\{-\cos\psi + \tfrac{1}{2}e(1+2\mathrm{m})\cos(\psi - f) + \tfrac{1}{2}e(1-2\mathrm{m})\cos(\psi+f) - 2e'\cos(\psi - f_1)\},$$

where

$$\alpha = a/a', \quad \psi = f + \varpi - f_1 - \varpi', \quad D = \alpha\, d/d\alpha, \quad \mathrm{m} = n'/n,$$

* The development to the second order with respect to the eccentricities and inclination is given by C. A. Shook, *Mon. Not. R.A.S.* vol. 91, p. 558.

and

$$(1-2\alpha\cos\psi+\alpha^2)^{-\frac{1}{2}}=\Sigma_i a_i \cos i\psi, \qquad i=0,\ \pm 1,\ \pm 2, \ldots.$$

Since
$$r\frac{\partial R}{\partial r}=a\frac{\partial R}{\partial a}, \qquad r=a(1-e\cos f),$$
we have

$$\begin{aligned}
r^2\frac{\partial R}{\partial r}=\alpha\frac{m'}{\mu}\Sigma_i\,&\{Da_i\cos i\psi-e(D+1-2\mathrm{m}i)\,Da_i\cos(i\psi+f)\\
&+e'(D+1-2i)\,Da_i\cos(i\psi+f_1)\}\\
+\alpha^2\frac{m'}{\mu}&\{-\cos\psi+e(1+\mathrm{m})\cos(\psi-f)\\
&+e(1-\mathrm{m})\cos(\psi+f)-2e'\cos(\psi-f_1)\},\\
-\frac{\partial}{\partial v}(q_0 r^2 R)=\alpha\frac{m'}{\mu}\Sigma_i i\,&\{a_i\sin i\psi-e(D+2-2\mathrm{m}i)\,a_i\sin(i\psi+f)\\
&+e'(D+1-2i)\,a_i\sin(i\psi+f_1)\}\\
+\alpha^2\frac{m'}{\mu}&\{-\sin\psi+\tfrac{1}{2}e(3+2\mathrm{m})\sin(\psi-f)\\
&+\tfrac{1}{2}e(3-2\mathrm{m})\sin(\psi+f)-2e'\sin(\psi-f_1)\}.
\end{aligned}$$

Substitutions from these formulae in 7·39 (1), (2), (3) give $\delta_0 q$, $\delta_0 u$, $n\delta_0 t$, and from the first term in each parenthesis the additional terms in 7·39 (4), (5). In most cases values of i beyond ± 4 will not be needed.

G. FINAL DEFINITIONS AND DETERMINATION OF THE CONSTANTS

7·41. The method of this chapter suggests the following definitions of the constants.

The mean motion n and the epoch ϵ are such that when all periodic and secular terms are suppressed, the true longitude shall be represented by $nt+\epsilon$. If terms dependent on the second powers of the masses be neglected there is no difference in the values of n, ϵ whether t or v be used as the independent variable, and the additions depending on these second powers will usually be insensible to observation.

The constants e_0, ϖ_0 have been defined to be such that the principal elliptic term in a_0/r, where $n^2 a_0^3 = \mu$, shall be represented by

$$\frac{e_0}{1 - e_0^2} \cos(v - \varpi_0).$$

Since any definition depends on the specification of some particular coefficient in a particular coordinate, a change to another definition can always be made. In numerical work it is usually sufficient to make any small correction due to an altered definition in the elliptic terms only.

The constants I_0, θ_0 are defined above by making the principal term in the latitude equal to $\sin I_0 \sin(v - \theta_0)$. Here I is the inclination of the two orbital planes. The change to i, the inclination to any other plane of reference, is made by the formulae of 1·32, and the change to i_0 is made to correspond. The slight difference when t is used as the independent variable will be sufficiently accounted for in the determination of the constants from observation.

7·42. A process of approximation is used in the determination of the values of the constants from observation. The perturbations may be calculated with the osculating elements at some given date unless previous work has given elements more nearly approximating to the constants of the theory. Thus constructed, the theory is compared with the observations, or with a selection from them. The differences are assumed to be due to erroneous values of the elements and are analysed so as to determine their corrections. While the formulae for the perturbations should be examined to see whether these corrections make any sensible difference in them, it will usually be found sufficiently accurate to correct the elliptic terms only.

7·43. The detailed work connected with the determination of the constants, as well as their correct definitions, has to be carried out whatever method be used to calculate the perturbations. The difficulty of avoiding error in performing the work can to some extent be lessened by

carrying it out as far as possible in a systematic way, since few checks of its accuracy are available. Most of the work of developing the disturbing forces can be done by harmonic analysis in the manner explained in 7·7, and this work has the advantages of being easily systematised and of carrying its own checks. The integration of the equations cannot be done in this manner, but the steps with the method of this chapter are easy and simple. The final step of comparing the calculated results with observation, although dismissed here in a few sentences, is or may be as laborious as that of calculating the general perturbations, but it is necessary if good values of the constants are to be obtained.

CHAPTER VIII

RESONANCE

8·1. Resonance is usually defined as a case of motion in which a particle or body, moving or capable of moving with periodic motion, is acted on by an external force whose period is the same as that of the motion of the body. This definition, while it describes the apparent character of the phenomenon, implies the existence of certain conditions which are not present in actual mechanical systems.

Let us take the usual illustration, namely the equation

$$\frac{d^2x}{dt^2} + n^2 x = m \sin n't.$$

When $n \neq n'$, we have the solution

$$x = c \sin (nt + \alpha) + \frac{m}{n^2 - n'^2} \sin n't.$$

But when $n = n'$, the solution is

$$x = c \sin (nt + \alpha) - \tfrac{1}{2} mn't \cos n't,$$

where, in both cases, c, α are arbitrary constants.

The illustration is defective because such an equation does not arise in any actual mechanical system except as an approximation, and because the approximation is valid only when x is small. The solution, therefore, breaks down as soon as $n - n'$ becomes too small. In actual mechanical problems, either the left-hand member which, equated to zero, gives the undisturbed motion, is not a linear function of x, or else the variable x is present in the expression for the disturbing forces, or both of these conditions may be present.

8·2. In the previous chapters we have based our procedure on the plan of continued approximation with respect to the disturbing mass. In the elliptic approximation this mass was neglected. In the first approximation to the disturbance the

elliptic values were substituted in the expressions for the disturbing forces, and the equations were again integrated. In the second approximation, the new values were substituted for the coordinates in the disturbing forces and the equations were again integrated. This procedure carried the implication that it was possible to develop the perturbations in positive integral powers of the disturbing mass, and that the coordinates would be expressed as sums of periodic terms. It is true that terms with coefficients increasing with the time were admitted, but it was seen that this was merely a convenient device adopted in order to abbreviate the calculations when the results were needed for a limited interval of time only. The terms so treated had periods which were long in comparison with the interval during which the expressions were to be used for comparison with observation.

In cases of resonance, this procedure fails. The reasons for its failure may be exhibited in several ways. That which is most fundamental in the mathematical development is due to the fact that expansions in powers of the disturbing mass have to be replaced by expansions in powers of the square root or of some other fractional power of this mass. Further, there is a fundamental discontinuity in the passage from non-resonance to resonance, which cannot be bridged by any mathematical device, since it is a physical characteristic of the motion.

The principal features of certain of the resonance problems in celestial mechanics can be illustrated by the motion of a pendulum which can make complete revolutions about a horizontal axis as well as oscillate about the vertical, and a following section (8·5) contains an analysis of these motions made from the point of view needed later.

8·3. We shall be concerned mainly with those cases of resonance which occur in the present configuration of the solar system. A certain number of such cases are present in the satellite systems of Jupiter and Saturn, where the periods of revolution round the planets appear to be very nearly in the ratio of two small integers. In the planetary system, we have the Trojan group of asteroids whose members circulate round the sun with

the same period as Jupiter. The motions of this group are treated in the following chapter. The most difficult problem is, however, to find out why, amongst the numerous asteroids which circulate between the orbits of Mars and Jupiter, none are known having periods exactly twice or three times that of Jupiter, or periods in the ratio to that of Jupiter of two small integers. A discussion of certain features of this problem will be given in this chapter.

8·4. In the actual cases of observed motions in the solar system, so far as they have been developed, we know of no case in which the discontinuity referred to in 8·2 is present in an observable form. We have referred to resonance as a set of cases in which the periods of revolution are in the ratio of two *small* integers. Since the final expressions for the coordinates contain all multiples of the frequencies, each pair of these can be regarded as a possibility for resonance conditions. But these frequencies are observed quantities, namely, those of the mean periods of revolution, and since such a pair of observed quantities can always be expressed as the ratio of two integers, it would seem that resonance must always be present in any three body problem.

The question goes further than this. It will appear below that the phenomena of resonance occur not only when the observed periods are *exactly* in the ratio of two integers but also when these periods are *nearly* in such a ratio. In other words, resonance occurs not only for a pair of special values of the periods but also *for a range of values and this range is finite.* One difficulty, namely, the question of the accuracy of our measures of the periods, disappears to some extent, but it is replaced by another, namely, the consideration of the infinite number of periodic terms which must have the resonance property.

The discontinuity referred to is not a place where either the coordinates or the velocities are discontinuous in a physical sense, but is one in which an infinitesimal change in one or more of the constants will ultimately produce a different type of motion. Thus the computer arrives at a situation where he needs a considerable increase in the accuracy with which the constants obtained from observation must be known in order to choose between two possible routes. And this process appears to continue as the approximations follow one another. From his point of view, there can be no general solution of the problem of three bodies, that is, there cannot exist one set of formulae giving the coordinates in terms of the time and the initial conditions which will serve for more than one set of such initial conditions, which will be valid for all time, and which can be used for calculation of the position. This conclusion may be a result of the mathematical devices which he adopts, but is more probably due to an inherent difficulty, namely, that of finding expressions which shall be continuous

functions of the constants which can be determined from observation. Any proper solution of the problem requires also the consideration of the limitations placed on the observer; it is not solely a mathematical problem.

8·5. *The motion of a pendulum.* The fundamental equation in resonance problems appears to be

$$\frac{d^2x}{dt^2} + \kappa^2 \sin x = 0. \qquad \qquad (1)$$

This is the same as the equation of motion of a simple pendulum of length l, if $\kappa^2 = g/l$, and if x be the angle which it makes with a vertical line drawn downwards at time t. Since the substitution, $x + \pi$ for x, changes the sign attached to κ^2, the equation with $-\kappa^2$ replacing κ^2 gives the same motion as (1).

The equation has the integral

$$\left(\frac{dx}{dt}\right)^2 = C + 2\kappa^2 \cos x, \qquad \qquad (2)$$

where C is an arbitrary constant: for the motion to be real it is necessary that $C + 2\kappa^2 \gtreqless 0$. There are three types of motion depending on $C > 2\kappa^2$, $C < 2\kappa^2$ and the intermediate case $C = 2\kappa^2$.

(i) $C > 2\kappa^2$. As dx/dt never vanishes in this case, it is always either positive or negative, and the pendulum is making complete revolutions in one sense or the other. We have

$$t = \int \frac{dx}{(C + 2\kappa^2 \cos x)^{\frac{1}{2}}} + \text{const.}$$

as the integral. If we put

$$\frac{1}{n} = \frac{1}{2\pi} \int_0^{2\pi} \frac{dx}{(C + 2\kappa^2 \cos x)^{\frac{1}{2}}},$$

n can replace C as the arbitrary constant, and the solution can be expanded in the form

$$x = nt + \epsilon + \frac{\kappa^2}{n^2} \sin(nt + \epsilon) + \frac{\kappa^4}{8n^4} \sin 2(nt + \epsilon) + \dots \qquad (3)$$

The periodic portion of this series can be regarded as an oscillation about the mean state of motion which is revolution with a period $2\pi/n$. The half-amplitude of this oscillation is evidently

less than π and it decreases as n increases. It is convenient to consider n, ϵ as the arbitrary constants of the motion to be determined from the initial conditions.

(ii) $C < 2\kappa^2$. Here $dx/dt = 0$ when $x = \pm\,\alpha$, where

$$\cos\alpha = -\,C/2\kappa^2.$$

The integral can be written in the form

$$\left(\frac{dx}{dt}\right)^2 = 4\kappa^2 \left(\sin^2 \tfrac{1}{2}\alpha - \sin^2 \tfrac{1}{2}x\right),$$

and x is a periodic function of t oscillating between the values $\pm\,\alpha$, where $\alpha < \pi$. The solution can be expanded into the series

$$x = c\sin(pt+\beta) + \frac{c^3}{192}\sin 3(pt+\beta) + \ldots, \qquad \text{......(4)}$$

where $$p = \kappa\left(1 - \tfrac{1}{16}c^2 + \ldots\right).$$

It is convenient here to consider c, β as the arbitrary constants, since the limit of p as c approaches zero is κ, a quantity independent of the arbitrary constants.

(iii) $C = 2\kappa^2$. Here $dx/dt = \pm\,2\kappa\cos\tfrac{1}{2}x$, the solution of which gives

$$x + \pi = 4\tan^{-1}\exp.(\kappa t + a_0), \qquad \text{...........(5)}$$

where a_0 is one arbitrary constant, the other having a particular value.

When $t = \pm\,\infty$, $x = \pm\,\pi$: at both places $dx/dt = 0$, $d^2x/dt^2 = 0$, and it follows by differentiation of (1) that all higher derivatives of x vanish. Near this point, while x approaches one of the limits $\pm\,\pi$, t tends to become an indeterminate function of x. It should be noted also that x is a discontinuous function of the arbitrary constant C, since the motion is of a quite different type according as $C \rightarrow 2\kappa^2$ from $C - 2\kappa^2 > 0$ or from $C - 2\kappa^2 < 0$. This result is of course characteristic of unstable equilibrium, but the point of view stated here is required in the applications to be made below.

Attention is drawn to the following facts which are obvious consequences, but which are needed for the interpretation of resonance equations.

(*a*) The mean value of dx/dt in (i) is n and in (ii) it is zero.

(*b*) As n passes from positive to negative through zero the solution given under type (i) is a discontinuous function of n at $n = 0$. With certain initial conditions, there is a *range of solutions* (independently of the time constant) corresponding to the case $n = 0$. This range of solutions constitutes type (ii) and is characterised by the constant c or α which is related to it, and α can have any value between $\pm\pi$.

(*c*) In case (i) the series giving the solution proceeds along powers of κ^2; in case (ii) it depends on $\sqrt{\kappa^2}$. There is no analytical continuity between the two types of solution, and they cannot be represented by one and the same analytic function of t.

(*d*) In case (i), the adopted arbitrary constants are the period of revolution and the time of passage through the vertical. In case (ii) they are the amplitude of the oscillation and time at which this oscillation vanishes.

8·6. A more general type of motion is exhibited by the equation

$$\frac{d^2x}{dt^2} + \frac{df(x)}{dx} = 0,$$

with its integral

$$\left(\frac{dx}{dt}\right)^2 = C - 2f(x),$$

where $f(x)$ is assumed to have an upper limit $f(\kappa)$. We get the same three types of motion according as $C > 2f(\kappa)$, $C < 2f(\kappa)$, $C = 2f(\kappa)$. In the first case x can take all its possible values; in the second case it is limited by the value given to C. In the first case also dx/dt never vanishes and it has a mean value different from zero; in the second case x is a periodic function of t and the mean value of dx/dt is zero. When $C = 2f(\kappa)$, $C - 2f(x)$ is divisible by $(x-\kappa)^2$ since κ is the value of x which makes $f(x)$ a maximum, so that dx/dt, d^2x/dt^2 and consequently all higher derivatives of x vanish.

8·7. *The disturbed pendulum.* The characteristics of resonance phenomena can be well exhibited by considering the equation

$$\frac{d^2x}{dt^2} + \kappa^2 \sin x = m\kappa^2 \sin(x - n't - \epsilon'), \quad \text{.........(1)}$$

which may be regarded as the equation of motion of a pendulum disturbed by a periodic force. We shall suppose that m, n', ϵ' are

given constants and that m is small compared with unity. We shall further suppose that when $m=0$, the pendulum is oscillating with a small amplitude, so that only the type (ii) with the solution 8·5 (4) is under consideration for the undisturbed motion.

To solve the equation (1) conveniently, it is advisable to use the method of the 'variation of arbitraries' The method is given in the following article for a more general type of equation than (1), as it serves to illustrate in detail the plan to be followed in cases where the undisturbed motion is periodic, and also the nature of the change of variables useful when resonance problems in celestial mechanics have to be considered.

8·8. It is proposed to find the solution of the equation

$$\frac{d^2x}{dt^2}+f'(x)=m\phi'(x,t), \quad \text{.................(1)}$$

when that of

$$\frac{d^2x}{dt^2}+f'(x)=0 \quad \text{.......................(2)}$$

is periodic and is known.

Suppose that the solution of (2) has been obtained in the form

$$x=x(l,c), \quad l=nt+\epsilon, \quad n=\text{func. } c, \quad \text{.........(3)}$$

where $x(l, c)$ is a Fourier series with argument l and with coefficients depending on c, the arbitrary constants being c, ϵ.

The solution (3) has the following properties. Let

$$x=x(l,c)$$

be regarded as a function of two independent variables l, c and let us form $n^2\partial^2x/\partial l^2$, substituting the result in

$$n^2\frac{\partial^2x}{\partial l^2}+f'(x). \quad \text{.......................(4)}$$

If n be the function of c defined by (3), the variables l, c disappear from (4) and the substitution reduces (4) to zero. The disappearance of l, c is not dependent on their *values*: they may be any functions of t or any variables whatever or constants.

Let us suppose that they are variable. We have

$$\frac{dx}{dt}=\frac{\partial x}{\partial l}\cdot\frac{dl}{dt}+\frac{\partial x}{\partial c}\cdot\frac{dc}{dt}.$$

We are about to replace x in (1) by two new variables l, c which are related to x by equations (3). Since we are replacing one variable by two others, a relation between the new variables is at our disposal. Let us so choose it that

$$\frac{\partial x}{\partial l}\cdot\frac{dl}{dt}+\frac{\partial x}{\partial c}\cdot\frac{dc}{dt}=n\frac{\partial x}{\partial l}, \qquad\qquad (5)$$

then

$$\frac{dx}{dt}=n\frac{\partial x}{\partial l}. \qquad\qquad (6)$$

Whence, since t is present in (6) only through l, c, n,

$$\frac{d^2x}{dt^2}=n\frac{\partial^2 x}{\partial l^2}\cdot\frac{dl}{dt}+\frac{\partial}{\partial c}\left(n\frac{\partial x}{\partial l}\right)\cdot\frac{dc}{dt}.$$

Substituting this in (1) and making use of the fact that (4) is zero, we have

$$n\frac{\partial^2 x}{\partial l^2}\cdot\left(\frac{dl}{dt}-n\right)+\frac{\partial}{\partial c}\left(n\frac{\partial x}{\partial l}\right)\cdot\frac{dc}{dt}=m\phi'. \qquad\qquad (7)$$

The equations (5) and (7) may be regarded as linear equations to find $dl/dt-n$, dc/dt. Their solution gives

$$\frac{dc}{dt}=\frac{m}{K}\frac{\partial x}{\partial l}\phi', \quad \frac{dl}{dt}=n-\frac{m}{K}\frac{\partial x}{\partial c}\phi', \qquad\qquad (8)$$

where

$$K=\frac{\partial}{\partial c}\left(n\frac{\partial x}{\partial l}\right)\cdot\frac{\partial x}{\partial l}-n\frac{\partial^2 x}{\partial l^2}\cdot\frac{\partial x}{\partial c}. \qquad\qquad (9)$$

It is easy to prove that K is a function of c only. For since the expression (4) vanishes identically for all values of l, c, its partial derivatives with respect to them will do so also. We thus obtain two equations between which $\partial f'(x)/\partial x$ can be eliminated and it is found that the result can be expressed in the form $\partial K/\partial l=0$, showing that K is independent of l and is therefore a function of c only.

If $\phi'=\partial\phi/\partial x$, where ϕ is a function of x, t, we can express (8) in the form

$$\frac{dc}{dt}=\frac{m}{K}\frac{\partial\phi}{\partial l}, \quad \frac{dl}{dt}=n-\frac{m}{K}\frac{\partial\phi}{\partial c}, \qquad\qquad (10)$$

where ϕ has been expressed in terms of l, c, t by inserting for x its value (3) in terms of l, c.

Since n, K are functions of c only, we can put (10) into canonical form with new variables c_1, B, defined by

$$dc_1 = Kdc, \quad dB = -n dc_1 = -nKdc,$$

and the equations then become

$$\frac{dc_1}{dt} = \frac{\partial}{\partial l}(B + m\phi), \quad \frac{dl}{dt} = -\frac{\partial}{\partial c_1}(B + m\phi). \quad \text{......(11)}$$

8·9. *Solution of the equations for l, c.* When m is small, the usual method of approximation is the substitution of constant values of n_0, ϵ_0, c_0 for n, ϵ, c in the terms factored by m which then become functions of t and can be integrated. If we put

$$l = l_0 + l_1 = n_0 t + \epsilon_0 + l_1, \quad c = c_0 + c_1,$$

and neglect powers of l_1, c_1 beyond the first, we have

$$\frac{dc_1}{dt} = m\left(\frac{1}{K}\frac{\partial\phi}{\partial l}\right)_0, \quad \frac{dl_1}{dt} = \left(\frac{\partial n}{\partial c}\right)c_1 - m\left(\frac{1}{K}\frac{\partial\phi}{\partial c}\right)_0, \quad \text{...(1)}$$

from which c_1 and then l_1 are immediately found.

In cases of resonance this procedure breaks down, and it is necessary to proceed as follows. Differentiate the equation 8·8 (9) for dl/dt and substitute the expressions for dc/dt, dl/dt in the result. We obtain

$$\begin{aligned}\frac{d^2l}{dt^2} &= \frac{\partial}{\partial c}\left(n - \frac{m}{K}\frac{\partial\phi}{\partial c}\right)\cdot\frac{dc}{dt} - \frac{m}{K}\frac{\partial^2\phi}{\partial l\partial c}\cdot\frac{dl}{dt} - \frac{m}{K}\frac{\partial^2\phi}{\partial c\partial t}\\ &= \frac{m}{K}\left(\frac{\partial n}{\partial c}\cdot\frac{\partial\phi}{\partial l} - n\frac{\partial^2\phi}{\partial l\partial c} - \frac{\partial^2\phi}{\partial c\partial t}\right)\\ &\quad + \frac{m^2}{K^2}\left\{\frac{\partial^2\phi}{\partial l\partial c}\cdot\frac{\partial\phi}{\partial c} - K\frac{\partial}{\partial c}\left(\frac{1}{K}\frac{\partial\phi}{\partial c}\right)\cdot\frac{\partial\phi}{\partial l}\right\} \quad \text{.........(2)}\end{aligned}$$

Since the last line of (2) has the factor m^2 it may, in general, be neglected in a first approximation.

In the applications, l, t are present in ϕ only as a sum of periodic terms with arguments $il - j(n't + \epsilon')$, where n', ϵ' are given constants. When this is the case

$$\frac{\partial\phi}{\partial t} = -\frac{j'n'}{i}\frac{\partial\phi}{\partial l},$$

and the first approximation to (2) can be written

$$\frac{d^2l}{dt^2}+\Sigma\,(in-jn')^2\frac{m}{iK}\frac{\partial}{\partial c}\left(\frac{1}{in-jn'}\,\frac{\partial\phi}{\partial l}\right)=0. \quad \ldots\ldots(3)$$

The standard type is that in which ϕ has the form

$$\phi=-a_i\cos l_i+b, \quad l_i=il-j\,(n't+\epsilon'), \quad \ldots\ldots\ldots(4)$$

where a_i, b are functions of c only. The equation for l_i is then

$$\frac{d^2l_i}{dt^2}+(in-jn')^2\frac{m}{iK}\frac{\partial}{\partial c}\left(\frac{a_i}{in-jn'}\right).\sin l_i=0. \quad \ldots\ldots(5)$$

If, in a first approximation, we put $c=c_0$, $n=n_0$, $K=K_0$—all constants—(5) takes the form of the equation for the pendulum. [If the coefficient of $\sin l_i$ be negative, we put

$$l_i=il-j\,(n't+\epsilon')+\pi$$

instead of the value (4).] There are therefore the types of solution considered in 8·5. Type (i) is that in which dl_i/dt is never zero so that in_0-jn' does not vanish. Type (ii) is that in which l_i oscillates about the value 0 [or π].

With type (i), we put $l_i=i\,(n_0t+\epsilon_0)-j\,(n't+\epsilon')=l_{i0}$ in the second term of (5) and deduce

$$l_i=l_{i0}+\frac{m}{iK}\frac{\partial}{\partial c}\left(\frac{a_i}{in-jn'}\right)\sin l_{i0}, \quad \ldots\ldots\ldots\ldots(6)$$

as a first approximation.

With type (ii), we choose n_0, ϵ_0 to be such that

$$in_0-jn'=0, \quad i\epsilon_0-j\epsilon'=0 \text{ or } \pi, \quad \ldots\ldots\ldots\ldots(7)$$

and l_i is an oscillating function. If the oscillations be small we can put $\sin l_i=l_i$, $n=n_0$, $c=c_0$, $K=K_0$ in order to find a first approximation. This gives

$$l_i=\lambda\sin(pt+\lambda_0), \quad p^2=\left|\frac{ma_i}{K}\left(\frac{\partial n}{\partial c}\right)_0\right|, \quad \ldots\ldots\ldots(8)$$

λ, λ_0 being arbitrary constants.

With similar limitations, the equation for c gives

$$\frac{dc}{dt}=\frac{m}{K}a_i\sin l_i=m\left(\frac{a_i}{K}\right)_0 l_i.$$

Whence $$c=c_0-m\left(\frac{a_i}{K}\right)_0\frac{\lambda}{p}\cos(pt+\lambda_0), \quad \ldots\ldots\ldots\ldots(9)$$

where c_0 is determined from $in_0 = jn'$, since n_0 is a known function of c_0.

The coefficient of the periodic term in (9) is

$$\left(\frac{ma_i}{K\partial n/\partial c}\right)^{\frac{1}{2}}\lambda, \qquad \dots\dots\dots\dots\dots\dots\dots(10)$$

and we thus have the first term of an expansion in powers of $m^{\frac{1}{2}}$. If the coefficient of $m^{\frac{1}{2}}$ is not large, the assumption that we can put $c = c_0$ in the coefficient of $\sin l$ in (5) is justified.

The difficult cases in celestial mechanics are those which depend on the value of c_0. If (10) becomes infinite as c_0 tends to zero, and if the coefficient of the periodic term in (9) is comparable with c_0, this method of approximation breaks down. The analogy of (5) with the pendulum equation no longer exists and special devices have to be employed in order to find out whether resonance is possible. A case of this kind in which $i = 1, j = 2$ is treated below.

In general, the solution (6) corresponds to the case of an ordinary perturbation and (8) to a case of resonance. The various features noted in 8·5 as peculiar to the two types of solution are present and can be interpreted in the light of our knowledge of the motion of the pendulum*.

8·10. *The general case of resonance in the perturbation problem.*

We recall the method of integrating the equations

$$\Sigma\,(dc_i\,.\,\delta w_i - dw_i\,.\,\delta c_i) = dt\,.\,\delta\,(\tfrac{1}{2}\mu^2 c_1^{-2} + mR), \quad \dots(1)$$

where mR now denotes the disturbing function, m being the disturbing mass with that of the sun as unit.

We had, with a slightly different notation,

$$R = R_0 + \Sigma A \cos j_1 N, \quad \dots\dots\dots\dots\dots\dots(2)$$

where A was a function of the c_i and

$$j_1 N = j_1 w_1 + j_2 w_2 + j_3 w_3 + j_1' w_1' + j_2' w_2', \quad \dots\dots(3)$$

$$w_1 = nt + \epsilon, \quad w_2 = \varpi, \quad w_3 = \theta, \quad w_1' = n't + \epsilon', \quad w_2' = \varpi'.$$

* A more elementary treatment of resonance with applications to the motions of one and two pendulums is given by E. W. Brown, *Rice Institute Pamphlets*, vol. XIX, No. 1. Also reprinted separately and issued by the Cambridge University Press.

In a first approximation we put $dw_1/dt = n_0$ and obtained integrals for the values of c_i, w_i which contained the divisors $j_1 n_0 + j_1' n'$. It was assumed that no one of these divisors vanished.

Let us now suppose that there is one term for which this condition does not hold, or rather, in order not to limit the argument too much, let us assume that there is one term in which neither j_1 nor j_1' is zero but in which $j_1 n_0 + j_1' n'$ is so small that the approximation is no longer valid, but that we can approximate with all the remaining terms. We shall see later on that this latter condition cannot hold, but that an approach to the solution can be made by supposing that it does hold.

All these remaining periodic terms can be eliminated by changes of variables in the manner explained in 6·6. We can therefore suppose that the equations (1) refer to the new variables after such terms have been eliminated and that

$$R = R_0 + A \cos j_1 N, \quad \text{.................}(4)$$

where $j_1 N$ has the value (3) and R_0 consists of those parts of R which are independent of w_1, w_1'.

Let us change the variable w_1 to W_1 where

$$j_1 W_1 = j_1 w_1 + j_1' w_1' + j_2' w_2', \quad \text{..............}(5)$$

so that*

$$j_1 \delta W_1 = j_1 \delta w_1, \quad j_1 d W_1 = j_1 dw_1 + j_1' n' dt. \quad \text{......}(6)$$

It is easily seen that the left-hand member of (1) merely requires the substitution of W_1 for w_1 if we replace the right-hand member by

$$dt \,.\, \delta \left[\frac{\mu^2}{2c_1^2} - \frac{j_1' n' c_1}{j_1} + mR \right]. \quad \text{..............}(7)$$

Next, replace c_1 by a new variable c_{11} defined by

$$c_1 = c_{10} + c_{11}, \quad j_1 \frac{\mu^2}{c_{10}^3} + j_1' n' = 0. \quad \text{...........}(8)$$

Previously, n_0 was defined for the case $m = 0$ by the relation $\mu^2 c_{10}^{-3} = n_0$, so that the second relation (8) is the same as $j_1 n_0 + j_1' n' = 0$. This definition does not demand that n_0 shall be

* The symbols d and δ have the same signification as in 5·3.

the final mean value of n, since there may be a constant portion in c_{11} which prevents this. The only condition needed at this stage is that c_{11}/c_{10} shall be small so that the expansion of c_1^{-2} in powers of this ratio shall be possible. We then have

$$\frac{\mu^2}{2c_1^2}-\frac{j_1'n'}{j_1}c_1=\frac{\mu^2}{2c_{10}^2}-\frac{j_1'n'}{j_1}c_{10}$$
$$-\left(\frac{\mu^2}{c_{10}^3}+\frac{j_1'n'}{j_1}\right)c_{11}+\frac{3}{2}\frac{\mu^2}{c_{10}^4}c_{11}^2+\dots \quad \dots\dots(9)$$

The first two terms of the right-hand member of (9) do not contain the variables and may therefore be omitted from (7); the coefficient of c_{11} vanishes in virtue of (8). Hence, inserting

$$n_0=\mu^2c_{10}^{-3}=-j_1'n'/j_1, \quad \dots\dots\dots\dots\dots(10)$$

we obtain for (7) the expression

$$dt\,.\,\delta Q=dt\,.\,\delta\left(\tfrac{3}{2}n_0\frac{c_{11}^2}{c_{10}}-\tfrac{4}{2}n_0\frac{c_{11}^3}{c_{10}^2}+\dots+mR_0+mA\cos j_1N\right). \quad \dots\dots(11)$$

The last expression is the characteristic form of the Hamiltonian function for cases of resonance. It is to be remembered that n_0, c_{10} are, by definition, functions of n' only and are therefore independent of c_i, W_1, w_i.

8·11. The equations for c_{11}, W_1 become

$$\frac{dc_{11}}{dt}=-mj_1A\sin j_1N, \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots(1)$$

$$\frac{dW_1}{dt}=-\frac{\partial Q}{\partial c_{11}}=-3n_0\frac{c_{11}}{c_{10}}+6n_0\frac{c_{11}^2}{c_{10}^2}-\dots-m\frac{\partial R_0}{\partial c_{11}}-m\frac{\partial A}{\partial c_{11}}\cos j_1N. \quad \dots\dots(2)$$

The right-hand members of the equations for c_2, c_3, w_2, w_3 all contain the factor m. If then we replace the variables c_{11}, c_2, c_3 by C_1, C_2, C_3, where

$$c_{11}=m^{\frac{1}{2}}C_1,\quad c_2=c_{20}+m^{\frac{1}{2}}C_2,\quad c_3=c_{30}+m^{\frac{1}{2}}C_3, \quad \dots(3)$$

and t by $m^{-\frac{1}{2}}T$, the equations can be written

$$\Sigma\,(dC_i\,.\,\delta W_i-dW_i\,.\,\delta C_i)$$
$$=dT\,.\,\delta\left(\tfrac{3}{2}n_0\frac{C_1^2}{c_{10}}-\tfrac{4}{2}n_0m^{\frac{1}{2}}\frac{C_1^3}{c_{10}^2}+\dots+R_0+A\cos j_1N\right), \quad \dots(4)$$

where W_2, W_3 are written for w_2, w_3 to preserve symmetry of form. The quantities c_{20}, c_{30} are now constants which are at our disposal. It will be noticed that the factor m has disappeared from the coefficient of $\cos j_1 N$.

In order to apply these results to actual problems, we need to know what the new variables mean in relation to the disturbed elliptic orbit. We have

$$c_1 = (\mu a)^{\frac{1}{2}}, \quad c_2 = c_1 \{(1-e^2)^{\frac{1}{2}} - 1\}, \quad c_3 = c_1 (1-e^2)^{\frac{1}{2}} (\cos i - 1).$$

The replacement of c_1 by $c_{10} + m^{\frac{1}{2}} C_1$, with the expansion in powers of $m^{\frac{1}{2}} C_1/c_{10}$, implies that we assume an initial major axis $2a_0$ and that its variations are small compared with $2a_0$. The factor $m^{\frac{1}{2}}$ would seem to imply that C_1 is not infinite when $m = 0$. Mathematically this is correct, but as the whole problem demands that m shall not be zero, we can at present be content with the previous statement.

Next, since in the problems considered $e < 1$, we have

$$c_2 = c_1 (-\tfrac{1}{2}e^2 - \tfrac{1}{8}e^4 - \ldots).$$

The replacement of c_2 by $c_{20} + m^{\frac{1}{2}} C_2$ implies that there is a value e_0 such that $(e - e_0) \div m^{\frac{1}{2}}$ is not very great. But care is necessary if we contemplate expansions in powers of $m^{\frac{1}{2}} C_2/c_{20}$. For perturbations by Jupiter, m/μ is of the order 10^{-3} so that $m^{\frac{1}{2}}$ is of order $\cdot 03\mu^{\frac{1}{2}}$. Thus expansions in powers of $(c_2 - c_{20})^{\frac{1}{2}}$ will converge too slowly for useful numerical computation if e_0 is much less than ·1. (See the last paragraph of 8·24.) The same difficulty does not occur in the case of c_3; for the expansion is made in powers of $(2 \sin \frac{1}{2}i)^2$, so that it involves positive integral powers only of $c_3 - c_{30}$. However, if we contemplate a development of R in powers of $m^{\frac{1}{2}}$, which these changes imply, we may be in danger of not obtaining a real approximation if $(2 \sin \frac{1}{2}i)^2$ is comparable with $m^{\frac{1}{2}}$.

These difficulties are actually present in the consideration of the motions of the asteroids circulating between the orbits of Mars and Jupiter. They play a much smaller part in the resonance cases amongst the satellites of Jupiter and Saturn, mainly because the disturbing mass-ratios are much smaller.

8·12. Let us suppose that expansions in powers of $m^{\frac{1}{2}}C_2/c_{20}$, $m^{\frac{1}{2}}C_3/c_{30}$ are possible, and let us further suppose that, in a first approximation to the solution of the equations 8·11 (4), we can neglect $m^{\frac{1}{2}}$.

The coefficient A then becomes a constant, A_0, and R is a function of w_2, w_3 only; thus C_2, C_3 disappear from Q and w_2, w_3 are therefore constant. The remaining equations are

$$\frac{dC_1}{dT} = -j_1 A_0 \sin j_1 N, \quad \frac{dW_1}{dT} = -3n_0 \frac{C_1}{c_{10}}, \quad \text{...........(1), (2)}$$

$$\frac{dC_2}{dT} = \frac{\partial R_0}{\partial w_2} - j_2 A_0 \sin j_1 N, \quad \frac{dC_3}{dT} = \frac{\partial R_0}{\partial w_3} - j_3 A_0 \sin j_1 N. \quad \text{......(3), (4)}$$

The first two equations give

$$\frac{d^2 W_1}{dT^2} - 3n_0 \frac{j_1 A_0}{c_{10}} \sin j_1 N = 0,$$

or, since w_2, w_3 are constant, so that $dN/dT = dW_1/dT$,

$$\frac{d^2 (j_1 N)}{dT^2} - 3\frac{n_0 j_1^2 A_0}{c_{10}} \sin j_1 N = 0. \quad \text{...........(5)}$$

This is the pendulum equation previously discussed. In the type of solution where $j_1 N$ makes complete revolutions so that dN/dt never vanishes, we have an ordinary perturbation; this was expressly excluded from the definition of N. In the second type $j_1 N$ oscillates about the value 0 or π according as A_0 is negative or positive. This oscillation is known as a *libration*.

In general, therefore, it appears that, under the stated conditions, such oscillations are possible. If the amplitude of the oscillation is small so that we may replace $\sin j_1 N$ by $j_1 N$ or by $j_1 N + \pi$, we have, after the replacement of dT by its value $m^{\frac{1}{2}} dt$,

$$j_1 N = \lambda \sin (pt + \lambda_0), \quad p^2 = 3j_1^2 \,|\, A_0 \,|\, \frac{n_0}{c_{10}} m, \quad \text{...(6)}$$

where λ, λ_0 are arbitrary constants.

The frequency p is proportional to the square root of the disturbing force, while the coefficient and phase are to be determined from observation. In all cases except that of the Trojan group

of asteroids in which $j_1 = -j'$, some power of e, e', Γ will be present in A and it is therefore necessary to consider the possibilities of expansions in powers of $e^{\frac{1}{2}}$, $e'^{\frac{1}{2}}$, $\Gamma^{\frac{1}{2}}$ as well as those in powers of $m^{\frac{1}{2}}$.

The value of C_1 is given by (2), (6). We find

$$c_1 = c_{10} + m^{\frac{1}{2}} C_1 = c_{10} - \frac{c_{10}}{3n_0 j_1} p\lambda \cos(pt + \lambda_0). \quad \ldots(7)$$

The small factor p in the coefficient of the periodic part of c_1 is consistent with the assumption, made earlier, that expansions in powers of $m^{\frac{1}{2}} C_1/c_{10}$ are possible. It shows further that *while the libration of N, that is of the angular position of the body, may have a finite amplitude, that of c_1 and therefore of the major axis is small.*

Since R_0, w_2, w_3 are constants, the integrals of (3), (4) are

$$C_2 = m^{\frac{1}{2}} t \frac{\partial R_0}{\partial w_2} + \frac{j_2}{j_1} C_1 + \text{const.}, \quad \ldots\ldots\ldots\ldots(8)$$

$$C_3 = m^{\frac{1}{2}} t \frac{\partial R_0}{\partial w_3} + \frac{j_3}{j_1} C_1 + \text{const.} \quad \ldots\ldots\ldots\ldots(9)$$

Now R_0 contains w_2, w_3, w_2' only in the form of cosines of multiples of $w_2 - w_2'$, $w_2 + w_2' - 2w_3$. In order therefore that C_2, C_3 shall not increase continually with the time, it is necessary that

$$w_2 = w_2' = w_3. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(10)$$

Since $\qquad j_1 N = j_1 w_1 + j_2 w_2 + j_3 w_3 + j_1' w_1' + j_2' w_2'$,

where the sum of all the j_i, j_i' is zero, the condition (10) gives

$$j_1 N = j_1 (w_1 - w_2) + j_1' (w_1' - w_2'). \quad \ldots\ldots\ldots(11)$$

If, however, $e' = 0$, R_0 is a function of the c_i only, so that the condition (10) is not needed and as w_2' disappears we have

$$j_1 N = j_1 w_1 + j_1' w_1' + j_2 w_2 + j_3 w_3, \quad \ldots\ldots\ldots(12)$$

where, as usual,

$$j_1 + j_1' + j_2 + j_3 = 0.$$

Since the value of Q in 8·10 (11) does not contain the time explicitly, the integral $Q = \text{const.}$ exists. We have made no direct use of this integral in the investigation just given. This

omission corresponds to that in the case of the pendulum making small oscillations where it is more convenient to solve the equation directly than through the medium of its first integral.

8·13. *The constants.* The general solution of the equations of motion requires the presence of six arbitrary constants. When the libration of $j_1 N$ is zero, and $e' = 0$, the constants present in w_1, w_2, w_3 are all determinate since w_2, w_3 are given by 8·12 (10) and that in w_1 by the condition that $j_1 N$ must be zero or π. The constant c_{11} is determined by 8·10 (8). Thus the constants c_{20}, c_{30} only are at our disposal. But as the libration in general will exist and as its presence introduces two new arbitrary constants, the loss of four arbitrary constants is reduced to a loss of two. Since the two conditions 8·12 (10) disappear when $e' = 0$, there is no loss in this case. Thus when $e' = 0$, there is a finite range of values for each of the arbitrary constants: in other words, the resonance cases are not particular solutions, but are merely types of solutions in which all the arbitrary constants have finite ranges.

When $e' \neq 0$, the question of the ranges of the constants cannot be settled by the approximation used above: this involved the neglect of terms factored by m but the retention of those factored by $m^{\frac{1}{2}}$. The conditions 8·12 (10) may be merely limiting values about which oscillations can exist in the same manner that $N = 0$ is a limiting value about which librations are possible. The treatment of this case for the Trojan group will be found in Chap. IX.

8·14. It is evident that the change of variable, w_1 to W_1, eliminates t from all the angles for which the ratio j_1'/j_1 is the same: all these terms have in fact the resonance property and should properly be included with the single term chosen above.

After the change of variable, the Hamiltonian function does not contain the time explicitly and there is an integral of the equations, namely,

$$\frac{\mu^2}{2c_1^2} - \frac{j_1' n'}{j_1} c_1 + mR = \text{const.}$$

The succeeding change from c_1 to c_{11} gives, by 8·10 (9),

$$\frac{\mu^2}{2c_{10}^2}\left(3\frac{c_{11}^2}{c_{10}^2} - 4\frac{c_{11}^3}{c_{10}^3} + \ldots\right) + mR = \text{const.}$$

This equation may be regarded as determining c_{11} in terms of the remaining variables. A comparison with 8·5 (2) will show that c_{11} plays a rôle similar to that of dx/dt in the integral for the motion of the pendulum, and that the presence of resonance depends on the value attributed to the constant.

It should be pointed out that the investigation given in the preceding articles does not prove the existence of resonance; it merely shows that so far no condition preventing resonance has appeared.

The illustration afforded by the motion of the pendulum must be regarded as showing only the general nature of the problem. Difficulties from which it is free appear as soon as we begin to consider even the simplest case of actual resonance in the solar system. Some of these arise from the fact that the consideration of a single resonance term is not sufficient. For example, in the case of the 2 : 1 ratio, the principal terms present in the disturbing function are

$$A_1 e \cos(w_1 - 2w_1' + w_2), \quad A_1'e' \cos(w_1 - 2w_1' + w_2').$$

In the ordinary planetary theory, the variation of w_2' and especially its secular part can be neglected in a first approximation and the result may be later corrected sufficiently to satisfy the needs of observation. If, however, the former angle is oscillating about a mean value, it is necessary to consider the nature of the motion of the latter according as it oscillates or makes complete revolutions.

Another difficulty not exhibited by the pendulum is the existence of types of motion in which small oscillations do not exist but in which oscillations of finite amplitude can exist. In certain simple cases these types may be dealt with by the use of the periodic orbit and of variations from this orbit. But these methods have heretofore given little or no information as to the range of the oscillations and this range may be of importance in actual problems. If, for example, the eccentricity of an asteroid can become so large under the influence of Jupiter's attraction that its orbit can intersect that of Mars, it is only a question of time until a close approach to that planet will occur and such a close approach may alter the orbit so fundamentally that a completely new investigation of its further motion will be needed. A method of approach to the investigation of such cases is given below.

General Method for Resonance Cases

8·15. Certain features of resonance problems have been developed in the previous sections of this chapter. In this and the following sections a method of procedure applicable to certain of the actual cases of resonance in the solar system will be given.

The integers j, j', for which $j_1 n_0 - j_1' n'$ is very small or zero, are usually less than 5. Here n_0, n' are observed mean motions whose ratio can be expanded into a continued fraction. If the successive convergents be formed, the practical cases of resonance are those in which a convergent with small numbers is so close to the ratio that the next convergent is a fraction with large integers. Since the order of the coefficient with respect to the eccentricities and inclination is $|j_1 - j_1'|$ (cf. 4·15), it follows that the coefficients corresponding to the higher convergents will be very small, and it will be assumed that their effects can be neglected in the limited intervals during which it is desired to obtain an approximation to the motion.

The terms for which $j_1 n_0 - j_1' n'$ is not very small or zero can be eliminated by the method of 6·6 and the resulting function therefore contains w_1, w_1' only in the combinations $p(j_1 w_1 - j_1' w_1')$, where p is a integer. Further, since the new terms produced by the elimination of the short period terms have the factor m^2, they may, in general, be neglected. Thus we can take as the Hamiltonian function

$$\frac{\mu^2}{2c_1^2} + mR = \frac{\mu^2}{2c_1^2} + mR_0 + m\Sigma A \cos pj_1 N, \quad \ldots\ldots(1)$$

where R_0 contains the terms in the elliptic development of R independent of w_1, w_1', and $j_1 N$ contains these variables only in the combination $j_1 w_1 - j_1' w_1'$, where j_1, j_1' are given integers. Any multiples of w_2, w_3, w_2' may also be present in the angles.

The substitution of W for w_1 defined by

$$j_1 W = j_1 w_1 - j_1' w_1',$$

similar to that of 8·10 (5), is made. The equations still remain canonical if to (1) we add the term $j_1' n' c_1 / j_1$.

In view of the relations $c_1 = (\mu a)^{\frac{1}{2}}$, $n^2 a^3 = \mu$, it is convenient to put

$$c_1 = c_0 (1 + z)^{-\frac{1}{3}}, \quad c_0 = (\mu a_0)^{\frac{1}{2}}, \quad n_0{}^2 a_0{}^3 = \mu, \quad n_0 = j_1' n'/j_1, \qquad \text{.........(2)}$$

so that
$$n = n_0 (1 + z), \qquad \text{.......................(3)}$$

and z is a variable which in stable motion must oscillate between limits which are small compared with unity.

With the notation (2), the function (1) with the additional term $j_1' n' c_1/j_1$ can be written

$$n_0 c_0 \{\tfrac{1}{2}(1+z)^{\frac{2}{3}} + (1+z)^{-\frac{1}{3}}\} + mR,$$

or, on expansion in powers of z,

$$\tfrac{3}{2} n_0 c_0 + n_0 c_0 (\tfrac{1}{6} z^2 - \tfrac{4}{27} z^3 + \ldots) + mR, \qquad \text{.........(4)}$$

and with z replacing c_1 we have

$$\frac{\partial}{\partial z} = -\tfrac{1}{3} c_0 (1+z)^{-\frac{4}{3}} \frac{\partial}{\partial c_1}.$$

As t is not present explicitly in R, the expression (4) equated to a constant is an integral of the equations. It may be written

$$z^2 - \tfrac{8}{9} z^3 + \ldots + 6mR/n_0 c_0 = C. \qquad \text{...........(5)}$$

The symbol z corresponds to $m^{\frac{1}{2}} C_1$ used in 8·11 and therefore has the factor $m^{\frac{1}{2}}$: it follows that C can be regarded as having the factor m.

This equation is analogous to the first integral in the motion of the pendulum. In a first approximation, it is assumed that constant values can be given in R to all the elements except W. Retaining only the lowest power of z, we have

$$\pm z = (C - 6mR/n_0 c_0)^{\frac{1}{2}}. \qquad \text{.................(6)}$$

This will be expected to furnish at least two types of motion depending on the value assigned to C. It will be shown in 8·17 that we can go a step further and include in the fundamental equation, which is that for W, terms depending on $m^{\frac{3}{2}}$.

8·16. Although we are concerned in this chapter with resonance cases, it is of some interest to apply 8·15 (6) to cases in which z is small but never zero for any value of t.

Suppose that mR contains a single periodic term denoted by $An_0c_0 \cos j_1 N$ and that we include the non-periodic portion mR_0 in C. As z does not vanish, we can expand 8·15 (6) in the form

$$\pm z = C^{\frac{1}{2}} \left\{1 - \frac{1}{2}\frac{6mA}{C} \cos j_1 N - \frac{1}{16}\left(\frac{6mA}{C}\right)^2 (1 + \cos 2j_1 N) + \ldots\right\}. \quad \ldots\ldots(1)$$

If n_{00} be the observed value of the mean motion, the definition of z gives for its mean value

$$\frac{n_{00}}{n_0} - 1 = \frac{j_1 n_{00} - j_1' n'}{j_1 n_0} = C^{\frac{1}{2}}\left(1 - \frac{1}{16}\frac{36m^2A^2}{C^2} + \ldots\right). \quad \ldots\ldots\ldots(2)$$

Since n_{00} is nearly equal to n_0, they can be interchanged in the coefficients of periodic terms. By hypothesis, the value of C given by (2) is small compared with unity. Thus the coefficient of $\cos j_1 N$ in (1) receives the small divisor $C^{\frac{1}{2}}$. Hence the periodic term in z is large compared with the term having the same argument in R.

The canonical equation for w_1, with the definitions of W, z, n_0 in 8·15, gives

$$\frac{dW}{dt} = n_0 z - m\frac{\partial R}{\partial c_1},$$

the second term of which can be neglected in comparison with the first.

Integrating, and making use of (1), we have

$$W = (n_{00} - n_0)\,t + \text{const.} - \frac{3mA}{Cj_1}\sin j_1 N - \frac{1}{32}\left(\frac{6mA}{C}\right)^2 \frac{1}{j_1}\sin 2j_1 N.$$

Thus the principal perturbation produced in the longitude by a term of long period is

$$-\tfrac{1}{2} j_1 \frac{6n_0{}^2 Am}{(j_1 n_{00} - j_1' n')^2} \sin j_1 N,$$

and there is also another long period term with argument $2j_1 N$ having a coefficient

$$-\tfrac{1}{8} j_1 (\text{coef. of } \sin j_1 N)^2,$$

a result in accordance with that obtained in 6·18 and also in 7·32.

8·17. *The equation for W.* With the definitions in 8·15, the canonical equation for W becomes

$$\frac{dW}{dt} = n_0 z - m\frac{\partial R}{\partial c_1}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Differentiating with respect to t, we obtain

$$\frac{d^2W}{dt^2} = n_0\frac{dz}{dt} - m\frac{\partial^2 R}{\partial c_1 \partial W}\left(n_0 z - m\frac{\partial R}{\partial c_1}\right) - m\frac{\partial^2 R}{\partial c_1{}^2}\frac{dc_1}{dt}$$
$$- m\Sigma\left(\frac{\partial^2 R}{\partial c_1 \partial c_i}\frac{dc_i}{dt} + \frac{\partial^2 R}{\partial c_1 \partial w_i}\frac{dw_i}{dt}\right),$$

where, in the last term, i has the values 2, 3. Since the derivatives of c_1, c_2, c_3, w_2, w_3 contain m as a factor, we obtain, on neglecting terms in this equation which are factored by m^2,

$$\frac{d^2W}{dt^2}=n_0\frac{dz}{dt}-n_0zm\frac{\partial^2 R}{\partial c_1 \partial W}. \quad \text{..............(2)}$$

But the first of equations 8·10 (1) with 8·10 (7) gives, on substituting for c_1 its value 8·15 (2),

$$-\tfrac{1}{3}c_0(1+z)^{-\frac{4}{3}}\frac{dz}{dt}=m\frac{\partial R}{\partial W}, \quad \text{..............(3)}$$

or, neglecting terms of order m^2,

$$\frac{dz}{dt}=-\frac{3}{c_0}(1+\tfrac{4}{3}z)\,m\frac{\partial R}{\partial W}. \quad \text{..............(4)}$$

Now R is a function of c_1. If we put therein $c_1=c_0(1-\frac{1}{3}z)$ and expand R in powers of z, we obtain

$$R=(R)_0-\tfrac{1}{3}z\left(\frac{\partial R}{\partial c_1}\right)_0+\ldots, \quad \text{..............(5)}$$

where the notation $(\quad)_0$ implies that c_0 has been substituted for c.

Hence, to the order $m^{\frac{3}{2}}$,

$$\frac{\partial R}{\partial W}=\left(\frac{\partial R}{\partial W}\right)_0-\tfrac{1}{3}z\left(\frac{\partial^2 R}{\partial c_1 \partial W}\right). \quad \text{..............(6)}$$

On combining this result with (2), (4), and noticing that $(\partial^2 R/\partial c_1 \partial W)_0$ disappears, we obtain

$$\frac{d^2W}{dt^2}+\frac{3n_0m}{c_0}(1+\tfrac{4}{3}z)\left(\frac{\partial R}{\partial W}\right)_0=0, \quad \text{...........(7)}$$

in which, to the order of the terms retained, we can put

$$z=\frac{1}{n_0}\frac{dW}{dt}.$$

Thus the variable c_1 has been eliminated as far as the order $m^{\frac{3}{2}}$ and we can write (7) in the form

$$\frac{d^2W}{dt^2}\left(1-\frac{4}{3}\frac{1}{n_0}\frac{dW}{dt}\right)+\frac{3n_0}{c_0}m\left(\frac{\partial R}{\partial W}\right)_0=0. \quad \text{......(8)}$$

It is to be remembered that the variables c_2, c_3, w_2, w_3 are still present in the last term of (8).

8·18. Another integral can be obtained when $e'=0$. Since the disturbing function is a function only of the differences of the angles w_1, w_2, w_3, w_1', w_2', we have

$$\frac{\partial R}{\partial w_1}+\frac{\partial R}{\partial w_2}+\frac{\partial R}{\partial w_3}+\frac{\partial R}{\partial w_1'}+\frac{\partial R}{\partial w_2'}=0. \quad\quad\text{(1)}$$

But by hypothesis the part of R which we are using contains w_1, w_1' only in the combinations $j_1 w_1 - j_1' w_1'$, so that

$$j_1'\frac{\partial R}{\partial w_1}+j_1\frac{\partial R}{\partial w_1'}=0. \quad\quad\text{(2)}$$

On changing the variable w_1 to W, the new disturbing function has the same properties. Hence, from (1) and (2),

$$\left(1-\frac{j_1'}{j_1}\right)\frac{\partial R}{\partial W}+\frac{\partial R}{\partial w_2}+\frac{\partial R}{\partial w_3}=-\frac{\partial R}{\partial w_2'}.$$

Thence, with the help of the canonical equations 8·10 (1), we obtain by integration

$$\left(1-\frac{j_1'}{j_1}\right)c_1+c_2+c_3=\text{const.}-m\int\frac{\partial R}{\partial w_2'}\,dt. \quad\quad\text{(3)}$$

If, in accordance with a previous notation, we put

$$c_2=c_1\left(-\tfrac{1}{2}e_1^2\right), \quad c_3=c_1\left(-\Gamma_1\right),$$

and make use of 8·15 (2), we obtain

$$(1+z)^{-\frac{1}{3}}\left(1-\frac{j_1'}{j_1}-\tfrac{1}{2}e_1^2-\Gamma_1\right)=\text{const.}-\frac{m}{c_0}\int\frac{\partial R}{\partial w_2'}\,dt. \quad\text{(4)}$$

When $e'=0$, R is independent of $w_2'=\varpi'$. The last term of (4) disappears, and the equation becomes an integral.

The Case $e'=\Gamma=0$

8·19. The variables c_3, w_3 disappear and the canonical system reduces to one with four variables. The differences of the angles w_1, w_1', w_2 are present in R and the ratio of the multiples of w_1, w_1' is fixed by the resonance condition. Hence a single angle N and its multiples are alone present in R.

The system admits of the two integrals 8·15 (5), 8·18 (4). The latter enables us to eliminate the variable c_2. From the former,

with the equation for dz/dt, we can eliminate N and thus obtain an equation giving dz/dt in terms of z. After the integration of this equation, giving z in terms of t, the remaining variables may be found without difficulty. The process thus described will be followed below but will be simplified by the omission of terms known to be small in comparison with those retained.

We assume that z, e^2, m are small compared with unity. The omission of higher powers gives $e_1^2 = e^2$, and from 8·18 (4),

$$\tfrac{1}{2}e^2 - \tfrac{1}{3}\left(1 - \frac{\dot{j}_1'}{\dot{j}_1}\right)z = \text{const.}$$

or

$$e^2 = E - \tfrac{2}{3}\left(1 - \frac{\dot{j}_1'}{\dot{j}_1}\right)z, \qquad \text{.................(1)}$$

where E is a constant.

Next, if $R = R_0 + \Sigma A_p \cos pj_1 N$, R_0, A_p are functions of c_1 or z and c_2 only, and by the use of (1) can be expressed as functions of z. Since R_0 contains only even powers of e, it can be expressed as a positive power series in z, and the constant part is all that need be retained, although the retention of z, z^2 creates little additional labour. The coefficient A_p has the form

$$A_p = n_0 c_0 e^J (a_0 + a_2 e^2 + \ldots), \qquad J = p\,|j_1 - j_1'|,$$

where $a_0, a_2, \ldots$ can be expressed in series of positive powers of z; the same limitations as those made in the case of R_0 permit us to retain the constant term only.

Finally, on the same basis, we put $p = 1$ and thus reduce R to a single periodic term and a constant portion.

With these limitations, we can put

$$R = \text{const.} + n_0 c_0 A e^J \cos j_1 N. \qquad \text{..............(2)}$$

On substitution of this in 8·15 (5), we can suppose that the constant part is included in C and thus obtain

$$z^2 = C - 6mAe^J \cos j_1 N. \qquad \text{.................(3)}$$

With the same limitations, 8·17 (4) becomes

$$\frac{1}{n_0}\frac{dz}{dt} = 3\dot{j}_1 mAe^J \sin j_1 N. \qquad \text{.................(4)}$$

The elimination of j_1N between (3), (4) with the help of (1) gives

$$\frac{4}{j_1^2}\left(\frac{dz}{n_0dt}\right)^2+(z^2-C)^2=36m^2A^2e^{2J}=36m^2A^2\left\{E-\tfrac{2}{3}\left(1-\frac{j_1'}{j_1}\right)z\right\}^J. \quad \text{......(5)}$$

This equation has the form $dz/dt=\{f(z)\}^{\frac{1}{2}}$ and gives t as a function of z. For values of J less than 5, the integral is of the elliptic type and the discussion of (5) or of its integral gives the chief characteristics of the motion.

For the cases of chief interest in the solar system, J is, in fact, less than 5, and the equation includes all these cases. The most serious limitation is that introduced by the assumption $e'=0$.

8·20. *Particular cases.* These are classified according to the values of J.

For $J=0$, we have $j_i=j_i'=1$. This case, that of the Trojan group of asteroids, is treated in detail in Chap. IX. It permits of numerous simplifications, but the development of R takes a quite different form.

For $J=1$, the ratio j_1/j_1' has the values 1/2, 2/3, 3/4, ..., in the cases of exterior bodies disturbing interior ones, and their inverses when interior are disturbing exterior bodies. For these ratios, the mean values of a/a' are ·64, ·76, ·82, ... It is doubtful whether the expansions are sufficiently convergent for numerical calculations beyond the ratio 4/5. The case 1/2 is discussed in detail in the following paragraphs.

The case $J=2$*, corresponding to the ratios 1/3, 3/5, 5/7, ..., is rather more simple than the case $J=1$, owing to the fact that only even powers of e are present in the formulae. This case also arises when we take into account the inclination of the orbit. This and the higher values of J are chiefly of interest in the applications to asteroids disturbed by Jupiter.

* See Charlier, *Mech. des Himmels*, Absch. (1); D B. Ames, *Mon. Not. R.A.S.* vol. 92, p. 542.

THE CASE $j_1 = 1,\ j_1' = 2$

8·21. *Change of scale.* If we put

$$\omega z, \quad \frac{1}{n_0 \omega} t, \quad \left(\frac{\omega}{3}\right)^{\frac{1}{2}} e, \quad \text{for } z,\ t,\ e \text{ respectively,}$$

where $\omega = (12A^2 m^2)^{\frac{1}{3}}$, the equations given in the preceding sections become, with the given limitations and with appropriate changes in E, C,

$$\frac{dW}{dt} = z, \quad 2\frac{dz}{dt} = e \sin N, \quad N = w_1 - 2w_1' + \varpi = W + \varpi,$$

$$z^2 = C - e\cos N, \quad e^2 = E + 2z,$$

$$4\left(\frac{dz}{dt}\right)^2 = E + 2z - (z^2 - C)^2. \quad \ldots\ldots\ldots\ldots\ldots(1)$$

For an asteroid disturbed by Jupiter with $n_0/n' = 2$, we have* $mA = -\,{\cdot}000716$. For the change of scale we have $\omega = {\cdot}0183$, so that an actual eccentricity ·1 has the value ·78 in the new scale. The values of the variables in the new scale are thus comparable with unity.

8·22. There are two problems. One, that dealing with the conditions under which N is an oscillating angle (resonance) or a revolving angle (non-resonance). The other, the conditions under which z can pass through the value zero. The latter is not, in the limited case here treated, strictly a resonance problem, but it becomes one when $e' \neq 0$ and it is applicable to the cases of the apparent absence of asteroids for which the osculating mean motion is exactly twice that of Jupiter.

Conditions to be satisfied.

(i) Since z measures the deviation of the major axis from a mean value, it must be an oscillating function and must therefore lie between finite limits which will be denoted by $s \pm d$: we shall choose d to be a positive number so that $s + d$ is the maximum value of z and $s - d$ the minimum.

* *Mon. Not. R.A.S.* vol. 72, p. 619.

(ii) The limiting values of z are given by $dz/dt = 0$, so that $s \pm d$ are two of the roots of

$$E + 2z - (z^2 - C)^2 = 0. \quad \dots\dots\dots\dots\dots\dots(1)$$

(iii) The left-hand member of (1) must be > 0 for all other values of z.

(iv) The convergence of the developments is doubtful if the actual eccentricity is greater than about ·3; this gives a limit 2·4 to the variable in the new scale.

(v) The conditions $s = \pm d$ separate the cases in which z can or cannot take the value zero according as $s > 0$ or $s < 0$.

(vi) The equation $e^2 = E + 2z$ gives

$$e\frac{de}{dt} = \frac{dz}{dt}, \quad e\frac{d^2e}{dt^2} + \left(\frac{de}{dt}\right)^2 = \frac{d^2z}{dt^2}.$$

It is necessary to have $d^2z/dt^2 \neq 0$ when $dz/dt = 0$, in order that equation 8·21 (1) shall give a determinate value of z for all values of t. It follows from the equations just written that e, de/dt cannot be zero simultaneously. Since e is not negative, it can vanish only if de/dt vanishes simultaneously: hence e is never zero.

Since $z = s \pm d$ are two of the roots of (1), it is easily deduced that

$$C = s^2 + d^2 - \frac{1}{2s}, \quad E = 4s^2d^2 + \frac{1}{4s^2} - 2s, \quad \dots(2), (3)$$

and that 8·21 (1) may be written

$$-4\left(\frac{dz}{dt}\right)^2 = \{(z-s)^2 - d^2\}\left\{(z+s)^2 + \frac{1}{s} - d^2\right\} \dots\dots(4)$$

These results give

$$e^2 = 4s^2d^2 + \frac{1}{4s^2} - 2s + 2z, \quad \dots\dots\dots\dots\dots(5)$$

so that the maximum and minimum of e are given by

$$e = \left|\, 2sd \pm \frac{1}{2s} \right|. \quad \dots\dots\dots\dots\dots\dots\dots(6)$$

8·23. The identities

$$(s+d)^2 = s^2 + d^2 - \frac{1}{2s} + \left(2ds + \frac{1}{2s}\right),$$

$$(s-d)^2 = s^2 + d^2 - \frac{1}{2s} - \left(2ds - \frac{1}{2s}\right),$$

compared with 8·22 (6) and with

$$z^2 = s^2 + d^2 - \frac{1}{2s} - e\cos N,$$

show that if $s > 0$, $d > 1/4s^2$, the extreme values of N are π, 0, but that if $d < 1/4s^2$, N takes the value π at both extremes. *Hence, the relation $d = 1/4s^2$ separates oscillating from revolving angles of N, that is, the resonance from the non-resonance case.* It is easily seen that the same statement is true if $s < 0$, but that in the latter case N takes the value zero at both extremes when $d = 1/4s^2$.

8·24. The four values of z given by 8·22 (4) when $dz/dt = 0$ are

$$z_1 = s + d, \qquad z_2 = s - d,$$

$$z_3 = -s + \left(d^2 - \frac{1}{s}\right)^{\frac{1}{2}}, \quad z_4 = -s - \left(d^2 - \frac{1}{s}\right)^{\frac{1}{2}}.$$

By hypothesis z is to lie between z_1, z_2. This condition demands that

$$z_1 > z_2 > z_3 > z_4,$$

or that

$$z_3 > z_4 > z_1 > z_2.$$

(*a*) For $s > 0$, $d^2 > 1/s$, the descending order of magnitude is z_1, z_2, z_3, z_4. The condition $z_3 < z_2$ gives

$$d < 2s, \quad d < s + \frac{1}{4s^2}.$$

(*b*) For $s > 0$, $d^2 < 1/s$, the roots z_3, z_4 are imaginary and dz/dt is always real between $z = z_1, z_2$. Hence for $s > 0$ the boundary is

$$d = s + 1/4s^2, \quad \text{when } d < 2s,$$

$$d^2 = 1/s, \quad \text{when } d > 2s.$$

If we regard s, d as the rectangular coordinates of a point on a curve, the two conditions are the equations of two bounding curves which meet and have the same tangent at $d = 2s = 2^{\frac{1}{3}}$.

(c) If $s < 0$, all four roots are real, and the ascending order of magnitude of the roots is z_2, z_1, z_4, z_3. This demands that $2s + d > 0$, $d < -s - 1/4s^2$. Since $d > 0$, $s < 0$ it is easily seen that the latter condition includes the former. It also requires that $s + d < 0$, and as we have $s - d < 0$, it follows that z does not change sign. *Hence, z cannot vanish for $s < 0$, that is*, if z can vanish its middle value is positive.

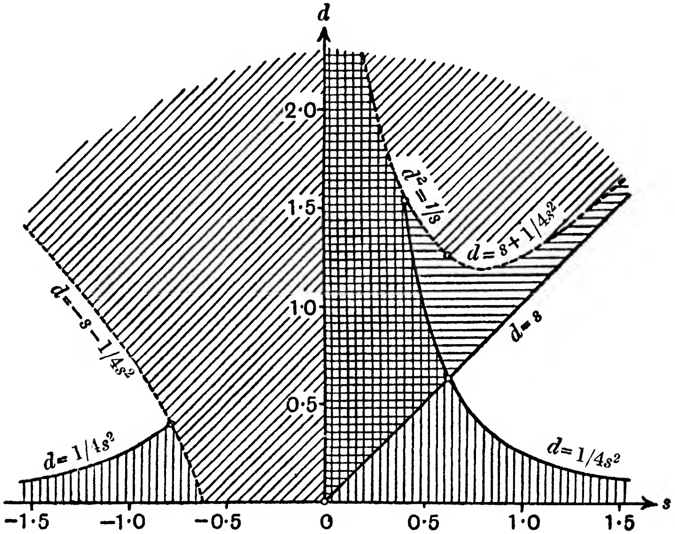

Fig. 2.

The conditions that z may vanish are therefore quite complicated. The boundary of the region consists of the four curves,

$$d = s, \quad d = s + 1/4s^2, \quad d^2 = 1/s, \quad s = 0,$$

and d, s are both positive.

On the other hand, the resonance regions for N are simply those portions included between the two curves

$$d = 0, \quad d = 1/4s^2$$

for $s < 0$ and $s > 0$, where z is real.

In Fig. 2, z is imaginary in the regions with inclined shading; z can be zero in the region with horizontal shading; N oscillates

in the region with vertical shading; the two latter regions overlap as shown.

Passage from $s > 0$ to $s < 0$ is effected when the minimum in the former case is the same as the maximum in the latter, that is, along the boundary $d = s + 1/4s^2$ for $s > 0$ to the boundary $d = -s - 1/4s^2$ for $s < 0$. Along these two curves and also along $d^2 = 1/s$, two of the values of z are equal and $W = \int z\,dt$ is indeterminate near $t = \pm\infty$ in the same sense as in the case of the pendulum near its highest position.

The only other case of equal roots for z is that given by $d = 0$. When $s > 0$ and d is small, the second factor of 8·22 (4) is approximately $4s^2 + 1/s$ which is constant. The solution is then

$$z = s + d\cos(qt + q_0), \quad q^2 = s^2 + 1/4s.$$

The same solution is available for $s < 0$ provided s is not too near the value given by $s^2 + 1/4s = 0$. In these cases the mean value of e is $1/2\,|s|$, so that s must not be too small. The case $d = 0$ in which z oscillates about the value s is the resonance case for $N = \pi$ when the libration is zero.

From 8·22 (2), (3), we deduce

$$C^2 - E = (s^2 - d^2)\left(s^2 - d^2 + \frac{1}{s}\right),$$

so that $C^2 - E$ changes sign at the boundary separating the region in which z can be zero from that in which it is never zero. But $C^2 - E = 0$ also when $d^2 = s^2 + 1/s$, a relation which does not enter into the discussion given above. It follows that the condition $C^2 = E$ is not the necessary and sufficient condition that z shall take the value zero*.

In vol. 4, chap. 25 of his *Mécanique Céleste*, Tisserand treats the resonance case by supposing that the eccentricity is equal to $e_0 + \delta e$, where e_0 is a constant, and he expands in powers of $\delta e/e_0$. With a proper choice of e_0 this is theoretically possible, since e is essentially positive, but it gives very slow convergence in the most important cases—those in which e_0 is small.

8·25. *Applications.* The discussion in the previous sections is applicable to the cases of asteroids whose mean motions are nearly twice that of Jupiter. The statistical discussions† show

* For a different and less complete discussion see E. W. Brown, *Mon. Not. R.A.S.* vol. 72, pp. 609–630.

† These have been numerous. Fairly complete lists are given by S. G. Barton, *A.J.* 702, 838; A. Klose, *Mitteil. Univ. Riga*, 1928.

that while there are numerous asteroids with mean motions somewhat greater and somewhat less than twice that of Jupiter, there is none which can be stated with certainty to have the relation satisfied within a certain range. This result refers to osculating elements. If we omit the short period terms, the variable z may be regarded as an element of this nature and the vanishing of z corresponds to the exact relation.

Now we have seen that the limiting case in which z can be zero is given by $s = d$ and the maximum value of z is then $2s$. According to 8·22 (6) the maximum value of e is then $2s + 1/2s$ and the least value which this expression can have is given by $s = \frac{1}{2}$. The least maximum of e is, therefore, 1·5. On referring back to the scale relation in 8·21, we find that this gives a least maximum for the eccentricity of ·13.

So far, therefore, nothing has been proved which prevents the existence of asteroids which can have an osculating mean motion exactly twice that of Jupiter. But it has been shown that if such asteroids can exist, the elements, in particular the eccentricity, must oscillate through a considerable range of values; small oscillations are impossible.

The existence of asteroids or satellites in which the angle N oscillates is a quite different question. What has been shown is that if such orbits exist, the middle value s of z must be different from zero. Small oscillations or librations about this value are possible. We have, for example, the case of Titan and Hyperion, satellites of Saturn, where the ratio is nearly 3:4, a case similar to that of 1:2.

The Cases $e' \neq 0$

8·26. These cases are much more difficult, mainly because the integral $e^2 = E - \frac{2}{3}(1 - j_1'/j_1)\, z$ no longer exists. But in the cases of the ratios $j_1' = j_1 + 1$, where the principal terms are of the first order with respect to the eccentricities, it is possible, in a first approximation, to utilise the results obtained above by a change of variables.

For simplicity let us consider the case $j_1 = 1$. The additional first order term in R has the form $n_0 c_0 e' A' \cos N'$, where

$N' = W + \varpi'$. Arguments similar to those used above give the equations for W, z:

$$z = \frac{1}{n_0}\frac{dW}{dt}, \quad \frac{1}{n_0}\frac{dz}{dt} = 3m\,(eA\sin N + e'A'\sin N'),$$

$$z^2 = C - 6mAe\cos N - 6mA'e'\cos N'.$$

Instead of the variables c_2, w_2, let us transform to the variables p_2, q_2 defined by

$$p_2 = e_1 c_1^{\frac{1}{2}} \sin w_2, \quad q_2 = e_1 c_1^{\frac{1}{2}} \cos w_2,$$

where, as before, $c_2 = -\frac{1}{2}c_1 e_1^2$. According to 5·14, the equations for c_1, W, p_2, q_2 are still canonical.

Let us change from p_2, q_2 to new variables defined by

$$p_2' = p_2 + \lambda e' \sin\varpi', \quad q_2' = q_2 + \lambda e' \cos\varpi',$$

where λ is a function of c_1 only. We have

$$\frac{dp_2'}{dt} = \frac{dp_2}{dt} + \frac{\partial\lambda}{\partial c_1}\cdot\frac{dc_1}{dt}e'\sin\varpi', \quad \delta p_2' = \delta p_2 + \frac{\partial\lambda}{\partial c_1}\delta c_1 . e'\sin\varpi',$$

with similar equations for dq_2'/dt, $\delta q_2'$.

Now $dp_2/dt = \partial R/\partial q_2$, and R is a linear function of p_2, q_2 for the only terms we have under consideration. It follows that dp_2/dt does not contain either eccentricity as a factor while dc_1/dt does contain them. The second term in the equation for dp_2'/dt is therefore two orders, with respect to the eccentricities, higher than the first term and may be neglected. The canonical equations may therefore be written

$$\begin{aligned} dt\,.\,\delta H &= dc_1\,.\,\delta W - dW\,.\,\delta c_1 + dp_2\,.\,\delta q_2 - dq_2\,.\,\delta p_2 \\ &= dc_1\,.\,\delta W - dW\,.\,\delta c_1 + dp_2'\,.\,\delta q_2' - dq_2'\,.\,\delta p_2' \\ &\qquad - \frac{\partial\lambda}{\partial c_1}(e'\cos\varpi'\,dp_2' - e'\sin\varpi'\,dq_2')\,\delta c_1. \end{aligned}$$

But the approximation $z = dW/dt$ to the equation $dW/dt = \partial H/\partial c_1$ involved the neglect of all parts of R in this equation and this is the only way in which the coefficient of δc_1 in the canonical set just given arises. It follows that the equations for c_1, W, p_2', q_2' are still canonical.

These results suggest that we can put

$$\bar{e}\cos\bar{\varpi} = e\cos\varpi + e'\frac{A'}{A}\cos\varpi',$$

$$\bar{e}\sin\bar{\varpi} = e\sin\varpi + e'\frac{A'}{A}\sin\varpi',$$

so that

$$\left.\begin{aligned} eA\sin N + e'A'\sin N' &= \bar{e}A\sin\bar{N}, \\ eA\cos N + e'A'\cos N' &= \bar{e}A\cos\bar{N}, \end{aligned}\right\} \quad \bar{N} = W + \bar{\varpi},$$

and that when we do so, $\bar{e}$, $\bar{N}$ will have the same properties that e, N had in the case $e' = 0$.

In particular, we shall have

$$\bar{e}^2 = E + \tfrac{2}{3}z, \quad \bar{e}^2 = e^2 + \frac{A'^2}{A^2}e'^2 + 2\frac{A'}{A}ee'\cos(\varpi - \varpi'),$$

and the limits previously given for e will now apply to $\bar{e}$.

In the cases of the asteroids disturbed by Jupiter we have $b'/b = -\cdot 36$, $e' = \cdot 048$ (*loc. cit.* 8·24), so that unless e is small the additional terms will not give large corrections to the results previously obtained as far as the vanishing of z is concerned.

In the cases of the small oscillations of N or N', it appears that these must take place about the values 0 or π and that $\varpi - \varpi'$ must oscillate in a similar manner. But the argument, based on the assumption that ϖ' is constant, is not necessarily valid if ϖ' has a mean motion.

8·27. The methods of this chapter are constructed mainly for the treatment of those cases of resonance which arise in the solar system. The theory of periodic orbits is applicable as a first approximation in certain problems: the asteroids which form the Trojan group are examples. In general, however, this theory fails, either because the numerical applications are too remote or because the restrictions under which the theory is developed avoid the very difficulties which the actual problems present.

The methods given above apply to cases of resonance in which both periods of revolution are present. The perihelia and nodes are angles which in general revolve and there are possibilities of resonance relations between their periods of revolution. In the comparatively short interval of time during which observations have been made, such relations are unimportant because, with the very long periods involved, expansions in powers of the

time give the required degree of accuracy. Comparable with these are the new periods introduced by the librations, and there are, therefore, further possibilities for resonance relations. So long as the past history of the solar system was supposed to be confined within an interval of 10^8 years, deductions as to its initial configuration from its present configuration appeared to have some degree of value; the extension of this interval to 10^9 years or longer makes these deductions quite doubtful. The doubt appears not so much in the ranges of values possible for the mean distances as in the ranges of the eccentricities and inclinations.

The indications furnished by the theory of resonance as applied to the solar system point towards the possibility of occasional large osculating eccentricities and inclinations at some time in the future. On the other hand, statistical evidence appears to indicate that these elements will tend to be confined within narrow limits. A discussion of these and other difficulties involved in the attempts to apply the theory to the solar system will be found elsewhere*.

* E. W. Brown, *Bull. Amer. Math. Soc.* May–June, 1928; *Publ. Astro. Soc. Pac.* Jan. 1932.

CHAPTER IX

THE TROJAN GROUP OF ASTEROIDS

9·1. *The triangular solutions of the problem of three bodies.*

The problem of three bodies does not, in general, admit a finite solution in terms of known functions. Laplace, however, has shown that there is a solution in which the three bodies always occupy the vertices of an equilateral triangle. The plane of the triangle is fixed and any two of the bodies describe ellipses having the same eccentricity about the third body which lies in a focus. Further, if n, a be the mean motion and semi-axis major of any one of these ellipses, the relation

$$n^2a^3 = \text{sum of the masses}$$

is found to be a necessary consequence of the solution. Other sets of finite solutions, in which the bodies are collinear, are known but they will not be considered here.

Small changes from the triangular configuration or in the appropriate velocities, or perturbations by other planets, cause oscillations about the triangular configuration, provided the masses satisfy a certain limiting configuration; the study of these oscillations and the applications of the theory are the objects of this chapter. Ten asteroids, each of which, with Jupiter and the Sun, apparently satisfies the given conditions, have been discovered, the first in 1901, more than a century after Laplace gave the solution, and the last in the year 1932. They have received names taken from the *Iliad* of Homer and from this circumstance constitute what is usually called the Trojan group.

We shall first prove the existence of the triangular solutions and of small oscillations of a certain kind about this solution; these will indicate some of the characteristic features of the motion. A general theory for the motion of an asteroid of the Trojan group will then be based on the methods used in Chaps. VI and VIII.

The problem differs from that of the ordinary planetary theory in several respects. In the first place, the development of the disturbing function given in Chap. IV cannot be used because the ratio of the mean distances of Jupiter and the asteroid is very near unity and that development ultimately depends on series in powers of this ratio which do not converge when the ratio is unity. In the second place, the motion is a case of resonance, since the ratio of the mean motions of the asteroid and Jupiter oscillates about the value unity. Thirdly, these oscillations, instead of being small, may have very considerable amplitudes and require special methods if an accuracy comparable with that of observation is to be secured. Another point, brought out in Chap. VIII, is the development in powers of the square root of the ratio of the mass of Jupiter to that of the Sun, instead of in integral powers of this ratio as in the ordinary planetary theory; since the square root of a small fraction is much greater than the fraction, the rate of numerical convergence may be much diminished in consequence. Still another peculiar feature is the theory of the long period terms produced by other planets, and notably by Saturn. A first approximation to their coefficients cannot be obtained by neglecting the action of Jupiter, and these coefficients tend to become greatest, not when the periods are longest, but when these periods approach most nearly to that of the principal libration.

9·2. *Existence of the triangular solutions.*

Since the motion takes place in a fixed plane, the latter may be used as the plane of reference. With the use of the equations of 1·23, those numbered (5), (6) disappear and $\mathrm{v} = v$. Let us take one of the bodies, mass m_0, as origin and let the coordinates and masses of the other two bodies be r, v, m_1 and r', v', m'. It is then necessary to show that the equations

$$\frac{d^2r}{dt^2} - r\left(\frac{dv}{dt}\right)^2 = \frac{\partial F}{\partial r}, \qquad \frac{d}{dt}\left(r^2\frac{dv}{dt}\right) = \frac{\partial F}{\partial v}, \quad \ldots(1), (2)$$

$$\frac{d^2r'}{dt^2} - r'\left(\frac{dv'}{dt}\right)^2 = \frac{\partial F'}{\partial r'}, \qquad \frac{d}{dt}\left(r'^2\frac{dv'}{dt}\right) = \frac{\partial F'}{\partial v'}, \quad \ldots(3), (4)$$

where, according to 1·9, 1·10, with the inclinations zero,

$$F = \frac{m_0 + m_1}{r} + m' \left\{\frac{1}{\Delta} - \frac{r \cos (v - v')}{r'^2}\right\}, \quad \text{.........(5)}$$

$$F' = \frac{m_0 + m'}{r'} + m_1 \left\{\frac{1}{\Delta} - \frac{r' \cos (v' - v)}{r^2}\right\}, \quad \text{.........(6)}$$

$$\Delta^2 = r^2 + r'^2 - 2rr' \cos (v - v'),$$

are simultaneously satisfied by $r = r' = \Delta$, $v - v' = \pm 60°$, with elliptic motion for each of the bodies.

According to 3·2 (1), (2), these conditions demand that

$$\frac{\partial F}{\partial r} = \frac{\partial F'}{\partial r'} = -\frac{\mu}{r^2}, \quad \frac{\partial F}{\partial v} = \frac{\partial F'}{\partial v'} = 0. \quad \text{............(7)}$$

From (5) we have

$$\frac{\partial F}{\partial r} = -\frac{m_0 + m_1}{r^2} - m' \left\{\frac{r - r' \cos (v - v')}{\Delta^3} + \frac{\cos (v - v')}{r'^2}\right\},$$

$$\frac{\partial F}{\partial v} = -m' \left(\frac{1}{\Delta^3} - \frac{1}{r'^3}\right) rr' \sin (v - v'),$$

with similar expressions for $\partial F'/\partial r'$, $\partial F'/\partial v'$. It is at once evident that the equations (7) are satisfied by the given relations provided $\mu = m_0 + m_1 + m'$.

The elements n, a, e are evidently the same for the two ellipses with $n^2 a^3 = m_0 + m_1 + m'$. For the remaining elements we have $\epsilon - \epsilon' = \varpi - \varpi' = \pm 60°$.

9·3. *The equations of variations.*

These equations are defined by giving to the coordinates in the general equations of motion small additions to their elliptic values, the squares, products and higher powers of these additions being neglected. This procedure is not sufficient for the calculation of the general perturbations, but it serves to indicate their nature to some extent. The actual calculation of the perturbations is more easily carried out by quite different methods.

The problem will be limited now and throughout the remainder of this chapter by supposing that the mass m_1 of one body (the

asteroid) is so small compared with either of the masses of the other two bodies (the Sun and Jupiter), that it can be neglected in the equations of motion. We then have $F' = (m_0 + m')/r'$ and the motion of m' relative to m_0 is elliptic with m_0 in one focus.

In the present section, two further limitations will be made. The motion of m' relative to m_0 will be supposed to be circular and to receive no disturbance, and the disturbance of m_1 will be supposed to take place within the plane of motion of m', so that the problem of the motion of m_1 is still two-dimensional.

According to these assumptions, the undisturbed motion of m_1 will be circular. Denote this motion by the suffix zero and the disturbed values by

$$r = r_0 + \delta r, \quad v = v_0 + \delta v.$$

Substitute these values in 9·2 (1), (2) and expand in powers ot δr, δv and of their derivatives, neglecting powers and products of these quantities above the first. Since the equations are satisfied when δr, δv and their derivatives are zero, this procedure gives

$$\frac{d^2}{dt^2}\delta r - \left(\frac{dv_0}{dt}\right)^2 \delta r - 2r_0 \frac{dv_0}{dt}\frac{d}{dt}\delta v = \left(\frac{\partial^2 F}{\partial r^2}\right)_0 \delta r + \left(\frac{\partial^2 F}{\partial r \partial v}\right)_0 \delta v,$$

$$\frac{d}{dt}\left(r_0^2 \frac{d}{dt}\delta v + 2r_0 \frac{dv_0}{dt}\delta r\right) = \left(\frac{\partial^2 F}{\partial r \partial v}\right)_0 \delta r + \left(\frac{\partial^2 F}{\partial v^2}\right)_0 \delta v.$$

These are the 'equations of variations.'

The second derivatives of F are formed from 9·2 (5), (6). They are

$$\frac{\partial^2 F}{\partial r^2} = \frac{2(m_0 + m')}{r^3} - \frac{m'}{\Delta^3} + \frac{3m'}{\Delta^5}\{r - r'\cos(v - v')\}^2,$$

$$\frac{1}{m'}\frac{\partial^2 F}{\partial r \partial v} = -\left(\frac{1}{\Delta^3} - \frac{1}{r'^3}\right) r' \sin(v - v') + 3rr'\frac{r - r'\cos(v - v')}{\Delta^5}\sin(v - v'),$$

$$\frac{1}{m'}\frac{\partial^2 F}{\partial v^2} = -\left(\frac{1}{\Delta^3} - \frac{1}{r'^3}\right) rr'\cos(v - v') + 3\frac{r^2 r'^2}{\Delta^5}\sin^2(v - v').$$

The limitations imposed above give

$$r_0 = a = r' = \Delta, \quad v_0 - v' = \pm 60°, \quad dv_0/dt = n, \quad m_1 = 0.$$

The substitution of these values in the second derivatives of F and in the equations of variations gives for the latter,

$$\frac{d^2}{dt^2}\delta r - 2an\frac{d}{dt}\delta v - n^2\delta r = \frac{8m_0 - m'}{4a^3}\delta r \pm \frac{3\sqrt{3}}{4a^2}m'\delta v,$$

$$a^2\frac{d^2}{dt^2}\delta v + 2an\frac{d}{dt}\delta r = \pm\frac{3\sqrt{3}}{4a^2}m'\delta r + \frac{9m'}{4a}\delta v.$$

These equations being linear with constant coefficients, their solution is obtained by assuming

$$\delta r = Ae^{\lambda t}, \quad \delta v = Be^{\lambda t},$$

where A, B, λ are constants. The substitution of these values gives, after division by $e^{\lambda t}$, the conditions

$$A\left(\lambda^2 - n^2 - \frac{8m_0 - m'}{4a^3}\right) - B\left(2an\lambda \pm \frac{3\sqrt{3}}{4a^2}m'\right) = 0, \quad \ldots(1)$$

$$A\left(2an\lambda \mp \frac{3\sqrt{3}}{4a^2}m'\right) + B\left(a^2\lambda^2 - \frac{9m'}{4a}\right) = 0. \quad \ldots(2)$$

The elimination of the ratio $A:B$ between these equations gives

$$\lambda^4 + \lambda^2\left(3n^2 - 2\frac{m_0 + m'}{a^3}\right) + \frac{9m'}{4a^3}\left(n^2 + \frac{2m_0 - m'}{a^3}\right) = 0.$$

The use of the relation $n^2a^3 = m_0 + m'$ and the introduction of m, where

$$m = \frac{m'}{m_0 + m'},$$

reduce this to

$$\lambda^4 + \lambda^2 n^2 + \tfrac{27}{4}n^4 m(1 - m) = 0.$$

If
$$27m(1 - m) < 1,$$

or $m <$ ·04 approximately, the roots are all pure imaginary and the motion is oscillatory. Since $m <$ ·001 in the case of the Trojan group, the condition is easily satisfied. If powers of m beyond the first be neglected, the roots are

$$\lambda = \pm n\iota\sqrt{\tfrac{27}{4}m}, \quad \pm n\iota(1 - \tfrac{27}{8}m), \qquad \iota = \sqrt{-1},$$

so that the periods are

$$2\pi \div n\sqrt{\tfrac{27}{4}m}, \quad 2\pi \div n(1 - \tfrac{27}{8}m).$$

With $2\pi/n = 11{\cdot}86$ years, $1/m = 1047$, the former period is 148 years and the latter nearly the same as that of revolution of the asteroid or of Jupiter round the Sun.

The oscillation having a long period is a first approximation to the effect usually known as the 'libration.' The short period oscillation will be seen below to correspond to the principal elliptic term in the motion of the asteroid, so that the principal part of the motion of the perihelion is $\frac{27}{8}mn$.

The ratio $B:A$ for the long period oscillation is given by either of the equations (1), (2) with $\lambda = n\iota\,(27m/4)^{\frac{1}{2}}$. From (2) we find

$$\frac{aB}{A} = \frac{\pm 3m' - 8a^3n^2\iota(\frac{27}{4}m)^{\frac{1}{2}}}{-27a^3n^2m - 9m'} = \mp \tfrac{1}{12} + \frac{\iota}{\sqrt{3m}},$$

with the aid of the relations $a^3n^2 = m_0 + m' = m'/m$. Since the second of the two terms is large compared with the first, the approximate ratio of $|aB|$ to $|A|$ is $1 : \sqrt{3m}$ or $18{\cdot}7 : 1$. As $|A|$ is the amplitude of the oscillation along the radius vector and $|aB|$ that perpendicular to it, it follows that the former is small compared with the latter, the ratio being nearly as $\sqrt{3m} : 1$.

The features of the motion brought out in this investigation, namely, the long period of the libration, the small disturbance along the radius vector as compared with that perpendicular to it, and the presence of $\sqrt{m}$, will be utilised in the general theory which follows*. Incidentally, it may be pointed out that they are present in all resonance problems occurring in connection with planetary motion, as can be shown from the results of Chap. VIII.

We now proceed to the general theory of the motion.

* The small oscillations were first fully treated by E. J. Routh, *Proc. Lond Math. Soc.* vol. 6 (1875); see also his *Dynamics of Rigid Bodies*, Part II, Chap. III. A special case of them is treated by Charlier, *Himmels-Mech.* vol. 2, Chap. IX.

GENERAL THEORY OF PERTURBATIONS DUE TO JUPITER

9·4. *The disturbing function.*

According to 1·10, the force-function for the action of a planet, mass m', on one of mass m_1 when the sun, mass m_0, is taken as the origin of coordinates, is

$$F = \frac{m_0 + m_1}{r} + m'\left(\frac{1}{\Delta} - \frac{r}{r'^2}\cos S\right), \quad \text{...........(1)}$$

where
$$\Delta^2 = r^2 + r'^2 - 2rr'\cos S, \quad \text{..............(2)}$$

S being the angle between the radii r, r'.

If S be eliminated between (1), (2), F can be written in the form

$$F = \frac{m_0 + m_1 + m'}{r} + R = \frac{\mu}{r} + R, \quad \text{...........(3)}$$

where
$$R = m'\left(\frac{1}{\Delta} - \frac{1}{r} + \frac{1}{2}\frac{\Delta^2}{r'^3} - \frac{1}{2}\frac{r^2}{r'^3}\right), \quad \text{...........(4)}$$

since the term $\frac{1}{2}m'/r'$, thus introduced, can play no part in the equations of motion which depend only on the derivatives of F with respect to the coordinates of m_1.

In the form (4), $R = 0$ when $r = r' = \Delta$. It can also be written

$$R = m'\left(1 - \frac{\Delta}{r'}\right)^2\left(\frac{1}{\Delta} + \frac{1}{2r'}\right) - m'\left(1 - \frac{r}{r'}\right)^2\left(\frac{1}{r} + \frac{1}{2r'}\right),$$

a form which at once shows that the first derivative of R with respect to any coordinate of m_1 vanishes when $\Delta = r = r'$.

The mass m_1 of the asteroid will be neglected in comparison with $m_0 + m'$. The osculating mean motions n, n' and mean distances a, a' will then be connected by the relations

$$n^2a^3 = n'^2a'^3 = m_0 + m' = \mu.$$

9·5. *The equations of motion.*

We start with the variables c_i, w_i (defined in 5·13 and used in Chap. VI) which satisfy the canonical equations,

$$\Sigma\,(dc_i\,.\,\delta w_i - dw_i\,.\,\delta c_i) = dt\,.\,\delta\left(\frac{\mu^2}{2c_1^2} + R\right). \quad \text{.........(1)}$$

Define a new variable τ by the relation

$$w_1 = n't + \epsilon' + \tau = w_1' + \tau, \quad \text{..............(2)}$$

and in (1) replace the variable w_1 by τ. Since $dw_1 = n'dt + d\tau$, $\delta w_1 = \delta\tau$, the equations for the variables c_i, τ, w_2, w_3 will still remain canonical if we add $n'c_1$ to R in the right-hand member of (1) so that the Hamiltonian function becomes

$$\frac{\mu^2}{2c_1^2} + n'c_1 + R. \quad \text{....................(3)}$$

Since all the c_i have the dimension $\sqrt{\mu a}$, and since a/a' is near unity, let us put

$$c_1 = c_1'(1+x) = \sqrt{\mu a'}(1+x), \quad c_2 = c_1'x_2, \quad c_3 = c_1'x_3. \text{...(4)}$$

The equations for the variables x, x_2, x_3, τ, w_2, w_3 still remain canonical and can be written

$$dx\,.\,\delta\tau - d\tau\,.\,\delta x + dx_2\,.\,\delta w_2 - dw_2\,.\,\delta x_2 + dx_3\,.\,\delta w_3 - dw_3\,.\,\delta x_3$$
$$= n'dt\,.\,\delta\left\{\frac{1}{2}\frac{1}{(1+x)^2} + 1 + x + \frac{a'R}{\mu}\right\}. \quad \text{.........(5)}$$

Since R is a homogeneous function of a, a' of degree -1 and contains m' as a factor, the Hamiltonian function is now a pure ratio like the variables x, x_2, x_3, τ, w_2, w_3, and $a'R/\mu$ has the factor $m = m'/\mu$.

It will be seen below that x consists of portions which have either the factor m or the factor $\tau_1\sqrt{m}$, where τ_1 is the coefficient of the principal term in the libration. No case of an asteroid is yet known* in which $|x|$ exceeds ·01, and as eccentricities and inclinations are in general of the order ·1, x will be treated as of the same order as the square of the eccentricities. We shall carry R as far as the fourth powers of the eccentricities and inclination for the long period terms and on this basis it is advisable to retain terms of orders x^4, mx^2. In such terms the eccentricities and inclination can be neglected, and then the retention of terms of order x^4, mx^2 presents no serious difficulty, but as it will much simplify the exposition to neglect them, they will not be retained in the developments which follow. In any

* **Except possibly Anchises discovered in 1931.**

case we are going to neglect terms in the Hamiltonian function which have the factor m^2, provided such terms have no small divisors, so that the neglect here of the terms of orders x^4, mx^2 follows a general rule.

With these limitations, the right-hand member of (5) can be expanded in powers of x, and becomes

$$n'dt\,.\,\delta\left(\frac{3}{2}x^2 - 2x^3 + \frac{a'\mathrm{R}}{\mu} + 2x\frac{a'^2}{\mu}\frac{\partial \mathrm{R}}{\partial a}\right) \text{ to order } m^{\frac{3}{2}}, \quad \ldots(6)$$

where R is the value of R when $a = a'$ or $x = 0$ and similarly for the definition of $\partial\mathrm{R}/\partial a$. The approximation $a = a'(1 + 2x)$ has been used in the expansion of R.

9·6. *Form of the expansion of R.*

As in Chap. IV, it is assumed that R is developable in powers of the eccentricities and mutual inclination. The development there given is available also here as far as the stage where it is reduced to odd powers of $1/\Delta_0$ (cf. 4·13), with

$$\Delta_0^2 = 1 + \frac{a^2}{a'^2} - 2\frac{a}{a'}(1 - \tfrac{1}{2}\Gamma)\cos(w_1 - w_1'),$$

and to derivatives of these powers with respect to a, w_1. With $w_1 - w_1' = \tau$, Δ_0 becomes a function of a, Γ, τ, and τ is an oscillating angle.

The angles present are g, g', $w_1 + w_1' - 2\theta$ or τ, w_2, w_3, w_1', ϖ'. Thus R consists of a sum of terms of the form

$$\Sigma K \cos N + \Sigma K' \sin N,$$

where K, K' are functions of x, x_2, x_3, τ and N of τ, w_2, w_3, t. The particular point to be remembered is that τ is present both in coefficient and angle because K, K' contain functions of Δ_0.

We shall distinguish between the terms containing w_1' after the substitutions, $g = w_1 - w_2$, $g' = w_1' - w_2'$, $w_1 = \tau + w_1'$, and those independent of w_1'. Our preliminary investigation showed that $d\tau/dt$ was small so that the terms in the former class have short periods and those in the latter have long periods or are constant. The latter class is also distinguished by being independent of t explicitly when the orbit of Jupiter is an ellipse.

These considerations give us the form of the development, but the actual development will be carried out by a method quite different from that of Chap. IV, chiefly because very considerable abbreviations of the work are possible with the use of the special properties which R possesses in the case of the Trojan group.

In general, it appears that we can secure sufficient accuracy for observational needs by taking the short period terms to the second order with respect to the eccentricities and inclination, and the terms independent of w_1' to the fourth order. Classified with respect to the arguments $g, g', w_1 + w_1' - 2\theta$ used in Chap. IV, the latter are

Arg. 0*, orders 0, e^2, e'^2, e^4, $e^2e'^2$, e'^4, Γ^2;

„ $g - g'$, orders ee', e^3e', ee'^3;

„ $2g - 2g'$, order $e^2e'^2$;

„ $2g - (w_1 + w_1' - 2\theta)$, order $e^2\Gamma$;

„ $g + g' - (w_1 + w_1' - 2\theta)$, order $ee'\Gamma$;

„ $2g' - (w_1 + w_1' - 2\theta)$, order $e'^2\Gamma$.

9·7. *Elimination of the short period terms.*

The plan adopted is that used in Chap. VI, namely, a change of variables which leaves the equations canonical. Owing, however, to the fact that the variable τ is contained in both coefficients and angles, the form of the transformation function S has to be modified. Further, use can be made of the fact that with the neglect of m^2, one variable x appears in the differential equations only in a linear form as shown in 9·5.

Let the short period terms be denoted by R_t and put

$$\frac{a'}{\mu}\mathrm{R}_t + x\frac{a'^2}{\mu}\frac{\partial \mathrm{R}_t}{\partial a} = \Sigma\,(K + xL)\cos N,$$

where, in accordance with 9·5 (6), K, L are independent of x, but are functions of x_2, x_3, τ, and

$$N = j'n't + \text{multiples of } \tau,\ w_2,\ w_3 + \text{const.},$$

as shown in 9·6.

* The term of order Γ is included in Δ_0.

Let us transform to new variables $x_0, x_{20}, x_{30}, \tau_0, w_{20}, w_{30}$ by means of the transformation function $\bar{S}$, where

$$\bar{S} = \tau_0 x + w_{20}x_2 + w_{30}x_3 - S,$$

$$S = \Sigma \frac{K_1 + xL_1}{j'n'} \sin N_0 + x\Sigma \left(\frac{M_1}{j'n'} \sin N_0 + \frac{Q_1}{j'n'} \cos N_0\right). \quad \ldots(1)$$

In this last expression,

K_1, L_1 are the values of K, L when τ_0 is put for τ;

M_1, Q_1 are functions of x, x_2, x_3, τ_0 to be determined;

N_0 is the value of N when τ_0, w_{20}, w_{30} are put for τ, w_2, w_3.

According to the general theory, the relations connecting the new and old variables are

$$\left.\begin{aligned} \tau &= \tau_0 - \frac{\partial S}{\partial x}, & x_0 &= x - \frac{\partial S}{\partial \tau_0}, & & \\ w_i &= w_{i0} - \frac{\partial S}{\partial x_i}, & x_{i0} &= x_i - \frac{\partial S}{\partial w_{i0}}, & i &= 2, 3, \end{aligned}\right\} \quad \ldots(2)$$

and the equations for the new variables will still be canonical provided we add $\partial\bar{S}/\partial t$ to the Hamiltonian function. From (1),

$$\partial\bar{S}/\partial t = -\partial S/\partial t,$$

and

$$\partial S/\partial t = \Sigma (K_1 + xL_1) \cos N_0 + x\Sigma (M_1 \cos N_0 - Q_1 \sin N_0). \quad \ldots(3)$$

In performing the transformation to the new variables, we shall neglect terms factored by m^2. We recall that x has the factor $m^{\frac{1}{2}}$ while S has the factor m. Hence, to the order $m^{\frac{3}{2}}$, from (2),

$$x^2 = x_0^2 + 2x_0 \frac{\partial S}{\partial \tau_0} = x_0^2 + 2x_0 \frac{\partial S_0}{\partial \tau_0}, \qquad x^3 = x_0^3, \quad \ldots(4)$$

where S_0 is the value of S when x_0, x_{20}, x_{30} are substituted for x, x_2, x_3 therein.

Next consider the portion of R independent of t explicitly, denoting it by R_c. Since the new variables differ from the old by terms having the factor m, we can put

$$\frac{a'}{\mu} \mathrm{R}_c + 2x \frac{a'^2}{\mu} \frac{\partial \mathrm{R}_c}{\partial a} = \frac{a'}{\mu} \mathrm{R}_{c0} + 2x_0 \frac{a'^2}{\mu} \frac{\partial \mathrm{R}_{c0}}{\partial a},$$

where terms factored by m^2 are neglected. (Incidentally, it may be noticed that the terms of order m^2 are all of short period, so that even if we were calculating the long period terms with terms of order m^2 retained in the Hamiltonian function, these could still be neglected.) Similarly, we can replace R_t by R_{t0}, and $\partial S/\partial t$ by $\partial S_0/\partial t$.

If then in (4) we put for S_0 its value deduced from (1), we obtain for the new Hamiltonian function expressed in terms of the new variables,

$$\tfrac{3}{2}x_0^2 - 2x_0^3 + \frac{a'}{\mu} R_{c0} + 2x_0 \frac{a'^2}{\mu} \frac{\partial R_{c0}}{\partial a}$$
$$+ 3x_0 \Sigma \left(\frac{1}{j'n'} \frac{\partial K_0}{\partial \tau_0} \sin N_0 + \frac{K_0}{j'n'} \frac{\partial N_0}{\partial \tau_0} \cos N_0 \right)$$
$$- x_0 \Sigma (M_0 \cos N_0 - Q_0 \sin N_0), \quad \ldots(5)$$

where, in all cases, the suffix zero denotes that the old variables are replaced by the new in the corresponding functions. The terms arising from R_{t0} have been cancelled by the same terms present in $\partial S_0/\partial t$, and terms factored by $x_0^2 m$ have been omitted.

Finally, if we determine the coefficients M_1, Q_1 by the relations

$$M_1 = \frac{3K_1}{j'n'} \frac{\partial N}{\partial \tau_0}, \quad Q_1 = -\frac{3}{j'n'} \frac{\partial K_1}{\partial \tau_0}, \quad \ldots\ldots\ldots\ldots(6)$$

relations which still hold to the required order when the suffix zero is inserted, the terms of order $m^{\frac{3}{2}}$ will disappear from (5) and it will be reduced to its first line.

The remaining portion of this chapter will be devoted to the determination of the new variables in terms of the time. After this work is done, the old variables will be obtained in terms of the time with sufficient accuracy if we substitute x_0, x_{20}, x_{30} for x, x_2, x_3 in S and its derivatives in equations (2).

9·8. *The expansion of R in powers of e, e′, Γ.*

We have, as in 4·1 (2) and 1·10 (2),

$$\Delta^2 = r^2 + r'^2 - 2rr' \cos S,$$
$$\cos S = (1 - \tfrac{1}{2}\Gamma) \cos (v - v') + \tfrac{1}{2}\Gamma \cos (v + v' - 2\theta).$$

If we put

$$\Delta_1^2 = 1 + \frac{r^2}{r'^2} - 2\frac{r}{r'}(1 - \tfrac{1}{2}\Gamma) \cos (v - v'),$$

the expansion of R given by 9·4 (4) in powers of the second term of $\cos S$ as far as the order Γ^2 can be put in the form

$$\frac{a'R}{\mu} = m\frac{a'}{r'}R_1 + \frac{1}{2}m\Gamma\frac{ra'}{r'^2}\left(\frac{1}{\Delta_1{}^3} - 1\right)\cos(v + v' - 2\theta)$$
$$+ \frac{3}{16}m\Gamma^2\frac{r^2a'}{r'^3}\cdot\frac{1}{\Delta_1{}^5}\{1 + \cos 2(v + v' - 2\theta)\}, \quad \text{...(1)}$$

where
$$R_1 = \frac{1}{\Delta_1} + \frac{1}{2}\Delta_1{}^2 - \frac{r'}{r} - \frac{1}{2}\frac{r^2}{r'^2}. \quad \text{..............(2)}$$

The long period terms defined in 9·7 will be calculated as far as the fourth order with respect to e, e', $\Gamma^{\frac{1}{2}}$, and the short period terms to the second order.

For this purpose, put

$$\frac{r'}{r} = \frac{a'}{a}(1 + \rho) = \alpha(1 + \rho),$$
$$v = w_1 + E, \quad v' = w_1' + E', \quad v - v' = \tau + \zeta,$$
$$R_0 = \frac{1}{\Delta_0} + \frac{1}{2}\Delta_0{}^2 - \alpha - \frac{1}{2}\frac{1}{\alpha^2}, \quad \text{..............(3)}$$
$$\Delta_0{}^2 = 1 + \frac{1}{\alpha^2} - \frac{2}{\alpha}(1 - \tfrac{1}{2}\Gamma)\cos\tau, \quad \text{...........(4)}$$

so that R_0, Δ_0 are the values of R_1, Δ_1 when the eccentricities vanish. Taylor's theorem then gives

$$\frac{a'R_1}{r'} = \frac{a'R_0}{r'} + \frac{a'\rho}{r'}\alpha\frac{\partial R_0}{\partial\alpha} + \zeta\frac{a'}{r'}\frac{\partial R_0}{\partial\tau} + \frac{1}{2}\frac{a'\rho^2}{r'}\alpha^2\frac{\partial^2 R_0}{\partial\alpha^2} + \dots, \quad \text{......(5)}$$

which is to be continued to the fourth powers of ρ, ζ for the long period terms.

For the calculation of the coefficients of the derivatives of R_0 in (5), we have, from 3·16,

$$\left.\begin{aligned} \frac{a}{r} &= 1 + (e - \tfrac{1}{8}e^3)\cos g + e^2\cos 2g, \\ E &= 2(e - \tfrac{1}{8}e^3)\sin g + \tfrac{5}{4}e^2\sin 2g, \end{aligned}\right\} \quad \text{............(6)}$$

with similar expressions for a'/r', E'. Since $g = w_1 + \varpi$, $g' = w_1' + \varpi'$, the long period terms will be those whose argu-

ments are multiples of $g-g'$. The expressions (6) are sufficient to obtain such terms to the fourth order, in spite of the omission of terms of the forms $e^4 \cos 2g$, $e^3 \cos 3g$, $e^4 \cos 4g$ in a/r, and of similar forms in E, a'/r', E'. For a term with argument $2g$ must be combined with one with arguments $2g$, $g+g'$, or $2g'$ to give terms of the required form and these have coefficients of the second order, so that the combination is of the sixth order. Similarly for the other terms omitted. Finally, R_0 and its derivatives have no short period terms, so that it is sufficient to omit such terms in the expansions of powers of ρ, ζ.

The advantage of this mode of development is seen by a reference to the results given in 9·9. Four of the coefficients are zero, three others are the same except for the numerical factors, two others have the same property, and two more differ only in the fourth order parts.

9·9. *The coefficients of the derivatives of R_0.*

To obtain these expressions put $a/r = 1 + u_1$, $a'/r' = 1 + u_1'$, so that

$$\rho \frac{a'}{r'} = \frac{a}{r} - \frac{a'}{r'} = u_1 - u_1'$$

has no long period term. The functions needed have the form

$$(u_1 - u_1')^i (r'/a')^{i-1} (E - E')^j,$$

where $i + j \leqq 4$. The calculations appear to be most easily carried out by expressing each such product as a sum of terms, each of the form PQ', where P is a function of u_1, E, and Q' of u_1', E', r'; from these products the terms independent of g, g', and those with arguments $g-g'$, $2g-2g'$, are easily selected. The positive and negative powers of r, r' which are needed and the positive powers of E, E' can be read off from Cayley's tables*.

Use can be made of the fact that $\rho a'/r'$, $E-E'$ both change sign when e, g are interchanged with e', g', so that terms of the forms $ee' \sin (g-g')$, $e^2e'^2 \sin 2(g-g')$ cannot be present in

* *Mem. Roy. Astr. Soc.* vol. 29, pp. 191–306.

products of ρ/r', ρ^3/r'^3 with ζ, ζ^3 (in the fourth order terms the divisor r' takes the value a'). The remaining terms have the form $(e^3e' - e'^3e)\sin(g-g')$, and these disappear on account of the relation between the coefficients of $\cos g$, $\sin g$ in a/r, E, respectively.

The following results for the terms independent of g, g' and for those dependent on $g-g'$ and its multiples, to the fourth order with respect to e, e', have been obtained:

$$\frac{\rho}{r'} = \frac{\rho\zeta}{r'} = \frac{\rho^3\zeta}{r'} = \frac{\rho\zeta^3}{r'} = 0,$$

$$\frac{a'\zeta}{r'} = (ee' - \tfrac{1}{8}e^3e' - \tfrac{1}{8}ee'^3)\sin(g-g') + \tfrac{5}{8}e^2e'^2\sin 2(g-g'),$$

$$2\frac{a'\rho^2}{r'} = (e^2 + e'^2 + \tfrac{3}{4}e^4 + \tfrac{1}{2}e^2e'^2) \\ - (2ee' + \tfrac{3}{4}e^3e' - \tfrac{3}{4}ee'^3)\cos(g-g') - \tfrac{5}{4}e^2e'^2\cos 2(g-g'),$$

$$\frac{1}{2}\frac{a'\zeta^2}{r'} = (e^2 + e'^2 + \tfrac{9}{64}e^4 + \tfrac{17}{64}e^2e'^2) \\ - (2ee' - \tfrac{7}{8}e^3e' - \tfrac{5}{8}ee'^3)\cos(g-g') - \tfrac{61}{32}e^2e'^2\cos 2(g-g'),$$

$$\frac{4}{3}\frac{a'\rho^3}{r'} = (e^4 + 2e^2e'^2) - (3e^3e' + ee'^3)\cos(g-g') \\ + e^2e'^2\cos 2(g-g'),$$

$$4\frac{a'\rho\zeta^2}{r'} = (e^4 - e'^4) - (2e^3e' - 2ee'^3)\cos(g-g'),$$

$$\frac{8}{9}\frac{a'\rho^2\zeta}{r'} = \frac{2}{21}\frac{a'\zeta^3}{r'} = (e^3e' + ee'^3)\sin(g-g') - e^2e'^2\sin 2(g-g'),$$

$$\frac{8}{3}\frac{a'\rho^4}{r'} = 2\frac{a'\rho^2\zeta^2}{r'} = \frac{1}{6}\frac{a'\zeta^4}{r'} = e^4 + e'^4 + 4e^2e'^2 \\ - (4e^3e' + 4ee'^3)\cos(g-g') + 2e^2e'^2\cos 2(g-g').$$

These are ready for substitution in 9·8 (5) which is the development of the first term of $a'R/\mu$ in 9·8 (1).

To the second order, we have

$$\frac{a'R_1}{r'} = R_0 + ee'\sin(g-g')\frac{\partial R_0}{\partial\tau} \\ + \{e^2 + e'^2 - 2ee'\cos(g-g')\}\left(\frac{\partial^2 R_0}{\partial\tau^2} + \tfrac{1}{4}\alpha^2\frac{\partial^2 R_0}{\partial\alpha^2}\right). \quad \ldots(1)$$

The calculation of the short period terms to the second order presents no difficulty. For the portion $m\,(a'/r')\,R_1$, we obtain

$$a'\rho/r' = e\cos g - e'\cos g' + e^2\cos 2g - e'^2\cos 2g',$$
$$a'\zeta/r' = 2e\sin g - 2e'\sin g' + \tfrac{5}{4}e^2\sin 2g - ee'\sin(g+g') - \tfrac{9}{4}e'^2\sin 2g',$$
$$2a'\rho^2/r' = -\tfrac{1}{2}a'\zeta^2/r' = e^2\cos 2g - 2ee'\cos(g+g') + e'^2\cos 2g',$$
$$a'\rho\zeta/r' = e^2\sin 2g - 2ee'\sin(g+g') + e'^2\sin 2g'.$$

Up to this point, no use has been made of the fact that a'/a is near unity, so that the development just given is quite general, at least as far as the second order. In order to deduce the results of 4·32, the expansions of the derivatives of R_0 in terms of the coefficients A_i and cosines of multiples of $w_1 - w_1'$ are to be substituted. In making the comparison, the difference in the definitions of the symbol α should be remembered. However, the expansion to the second order is not difficult whatever the method used; it is in the calculation of the terms of higher orders that the expressions become long and complicated, so that for them the method should be suited to the problem.

9·10. *Calculation of the derivatives of R_0.*

These derivatives can all be reduced to the calculation of derivatives with $\alpha = 1$. For, according to the definition in 9·5,

$$\alpha = \frac{a'}{a} = \frac{c_1'^2}{c_1^2} = (1+x)^{-2} = 1 - 2x + \dots,$$

and

$$R_0 = \left(R_0 - 2x\frac{\partial R_0}{\partial \alpha}\right)_{\alpha=1},$$

since it has been pointed out (9·7) that the first power of x is sufficient in the expansion of R. The last result still holds if we substitute for R_0 any one of its derivatives. Since we can neglect the fourth order terms in the coefficient of x, this coefficient will not need derivatives of R_0 beyond the third and the latter are already required in the calculation of the term independent of x.

For the calculation of the derivatives put

$$q = (1 - \tfrac{1}{2}\Gamma)\cos\tau, \quad Q^2 = 1 + \alpha^2 - 2\alpha q, \quad q_1 = (1 - \tfrac{1}{2}\Gamma)\sin\tau,$$

so that by 9·8 (3), (4),

$$R_0 = \frac{\alpha}{Q} + \frac{1}{2} - \frac{q}{\alpha} - \alpha.$$

Whence, when $\alpha = 1$, $Q^2 = 2(1-q)$,

$$R_0 = \frac{1}{Q} + \frac{1}{2}Q^2 - \frac{3}{2}, \quad \frac{\partial R_0}{\partial \alpha} = \frac{1}{2}\frac{1}{Q} - \frac{1}{2}Q^2,$$

$$\frac{\partial^2 R_0}{\partial \alpha^2} = -\frac{1}{Q^3} - \frac{1}{4}\frac{1}{Q} - 2 + Q^2, \quad \frac{\partial^3 R_0}{\partial \alpha^3} = \frac{3}{2}\frac{1}{Q^3} + \frac{3}{8}\frac{1}{Q} + 6 - 3Q^2,$$

$$\frac{\partial^4 R_0}{\partial \alpha^4} = \frac{9}{Q^5} - \frac{9}{2}\frac{1}{Q^3} - \frac{15}{16}\frac{1}{Q} - 24 + 12Q^2,$$

$$\frac{\partial R_0}{\partial \tau} = \left(-\frac{1}{Q^3} + 1\right) q_1, \quad \frac{\partial^2 R_0}{\partial \tau \partial \alpha} = \left(-\frac{1}{2}\frac{1}{Q^3} - 1\right) q_1,$$

$$\frac{\partial^3 R_0}{\partial \tau \partial \alpha^2} = \left(\frac{3}{Q^5} + \frac{1}{4}\frac{1}{Q^3} + 2\right) q_1,$$

$$\frac{\partial^4 R_0}{\partial \tau \partial \alpha^3} = \left(-\frac{9}{2}\frac{1}{Q^5} - \frac{3}{8}\frac{1}{Q^3} - 6\right) q_1,$$

$$\frac{\partial^2 R_0}{\partial \tau^2} = \frac{2}{Q^3} - \frac{1}{4}\frac{1}{Q} + 1 - \frac{1}{2}Q^2 - (\Gamma - \tfrac{1}{4}\Gamma^2)\frac{3}{Q^5},$$

$$\frac{\partial^3 R_0}{\partial \tau^2 \partial \alpha} = \frac{1}{Q^3} - \frac{1}{8}\frac{1}{Q} - 1 + \frac{1}{2}Q^2 - (\Gamma - \tfrac{1}{4}\Gamma^2)\frac{3}{2}\frac{1}{Q^5},$$

$$\frac{\partial^4 R_0}{\partial \tau^2 \partial \alpha^2} = -\frac{12}{Q^5} + \frac{7}{4}\frac{1}{Q^3} + \frac{1}{16}\frac{1}{Q} + 2 - Q^2 + (\Gamma - \tfrac{1}{4}\Gamma^2)\left(\frac{15}{Q^7} + \frac{3}{4}\frac{1}{Q^5}\right),$$

$$\frac{\partial^3 R_0}{\partial \tau^3} = \left\{-\frac{6}{Q^5} + \frac{1}{4}\frac{1}{Q^3} - 1 + (\Gamma - \tfrac{1}{4}\Gamma^2)\frac{15}{Q^7}\right\} q_1,$$

$$\frac{\partial^4 R_0}{\partial \tau^3 \partial \alpha} = \left\{-\frac{3}{Q^5} + \frac{1}{8}\frac{1}{Q^3} + 1 + (\Gamma - \tfrac{1}{4}\Gamma^2)\frac{15}{2}\frac{1}{Q^7}\right\} q_1,$$

$$\frac{\partial^4 R_0}{\partial \tau^4} = \frac{24}{Q^5} - \frac{5}{Q^3} + \frac{1}{16}\frac{1}{Q} - 1 + \frac{1}{2}Q^2 + (\Gamma - \tfrac{1}{4}\Gamma^2)\left(-\frac{120}{Q^7} + \frac{39}{2Q^5}\right) + (\Gamma - \tfrac{1}{4}\Gamma^2)^2\frac{105}{Q^9}.$$

9·11. *The additional portions of $a'R/\mu$ depending on Γ.*

The second term of 9·8 (1) has the factor Γ and the long period terms which it produces are of the fourth order at least, since the argument $v + v' - 2\theta$ must be combined with the arguments $2g$, $g + g'$, or $2g'$ to produce multiples of $w_1 - w_1'$, and these terms have the respective factors e^2, ee', e'^2.

In order to expand it, write $v + v' - 2\theta$ in the form

$$v - v' + 2v' - 2\theta.$$

It is then easy to prove that

$$\begin{aligned}\frac{1}{2}\frac{ra'}{r'^2}\left(\frac{1}{\Delta_1^3} - 1\right)\cos(v + v' - 2\theta)\\
&= -\frac{\partial}{\partial\Gamma}\left(\frac{a'R_1}{r'}\right)\cos(2v' - 2\theta)\\
&\quad + \frac{1}{2-\Gamma}\frac{\partial}{\partial\tau}\left(\frac{a'R_1}{r'}\right)\sin(2v' - 2\theta),\end{aligned}$$

where, as before, $a'R_1/r'$ is expressed as a function of τ, g, g'.

But, with the help of 9·8 (6),

$$\begin{aligned}\cos(2v' - 2\theta) &= \cos(2w_1' - 2\theta)\cos 2E' - \sin(2w_1' - 2\theta)\sin 2E'\\
&= \cos(2w_1' - 2\theta).(1 - 4e'^2 + 4e'^2\cos 2g')\\
&\quad - \sin(2w_1' - 2\theta).(4e'\sin g' + \tfrac{5}{2}e'^2\sin 2g'),\end{aligned}$$

and the only portions of this which will give long period terms as far as the fourth order including those with factor Γ are

$$\cos(2w_1' - 2\theta) - 2e'\cos(2w_1' - g' - 2\theta) + \tfrac{3}{4}e'^2\cos(2w_1' - 2g' - 2\theta). \quad \text{......(1)}$$

For $\sin(2v' - 2\theta)$, these cosines are changed to sines.

The first of the three terms of (1) gives long period terms by combination with the short period terms of $a'R_1/r'$ having arguments $2g$, $g + g'$, $2g'$, the second with those having arguments g, g', and the last with R_0. The derivatives with respect to Γ, τ present no difficulties, since they can be formed directly from the results in 9·10.

For the short period part which is taken to the second order only, we put $v = w_1$, $v' = w_1'$, $r = r' = a'$, $\Delta_1 = \Delta_0 = Q$.

The third term of $a'R/\mu$ in 9·8 (1), having the fourth order factor Γ^2, gives the single term $3\Gamma^2/16Q^5$.

The portions due to the factor $1 - \frac{1}{2}\Gamma$ have been retained throughout on account of the large numerical multipliers which accompany them. As in Chap. IV, their presence causes but little additional calculation, and adds considerably to the numerical convergence for large values of Γ. But it is easy if desirable to

expand the derivatives of R given in 9·10 in powers of Γ. For, when $a=1$, $\Gamma=0$,

$$\frac{\partial Q}{\partial \Gamma}=-\frac{1}{2}\frac{\cos\tau}{Q},\quad \frac{\partial^2 Q}{\partial \Gamma^2}=\frac{1}{4}\frac{\cos^2\tau}{Q^3},\ldots,$$

so that $$Q=Q_0-\frac{\Gamma}{2}\frac{\cos\tau}{Q_0}+\tfrac{1}{8}\Gamma^2\frac{\cos^2\tau}{Q_0^{\,3}}-\ldots, \qquad \ldots\ldots\ldots(2)$$

where $Q_0^{\,2}=2-2\cos\tau$.

9·12. *Transformation to the canonical elements.*

The element x_2 is given (cf. 9·5 (4) and 5·13 (1)) by

$$x_2=(1+x)\{(1-e^2)^{\frac{1}{2}}-1\}, \qquad \ldots\ldots\ldots\ldots\ldots(1)$$

from which e can be expressed in terms of $(-x_2)^{\frac{1}{2}}$ and x. But the canonical elements x_2, w_2 will be replaced later by p_2, q_2, where

$$p_2=e_1\sin w_2\,(c_1/c_1')^{\frac{1}{2}},\quad q_2=e_1\cos w_2\,(c_1/c_1')^{\frac{1}{2}}, \quad\ldots\ldots(2)$$

$$e_1=\{2-2(1-e^2)^{\frac{1}{2}}\}^{\frac{1}{2}}=e+\tfrac{1}{8}e^3+\ldots,\quad c_1/c_1'=1+x.$$

These correspond to the elements in 5·14.

As $w_2=\varpi$, and as we neglect powers of x beyond the first, these give

$$e\sin\varpi=p_2(1-\tfrac{1}{8}e^2-\tfrac{1}{2}x),\quad e\cos\varpi=q_2(1-\tfrac{1}{8}e^2-\tfrac{1}{2}x),$$

$$e^2=(p_2^{\,2}+q_2^{\,2})(1-x)-\tfrac{1}{4}(p_2^{\,2}+q_2^{\,2})^2,$$

which, with the relation $g-g'=\tau+\varpi'-\varpi$, permit us to express $a'R_1/r'$ in terms of these canonical elements with but little additional calculation.

For the terms containing Γ, θ, the substitutions

$$p_3=(2\Gamma)^{\frac{1}{2}}(1-\tfrac{1}{2}e^2+\tfrac{1}{2}x)\sin\theta,\quad q_3=(2\Gamma)^{\frac{1}{2}}(1-\tfrac{1}{2}e^2+\tfrac{1}{2}x)\cos\theta, \qquad \ldots\ldots(3)$$

can be further abbreviated by putting unity for $1-\frac{1}{2}e^2+\frac{1}{2}x$ except in the factor $1-\frac{1}{2}\Gamma$, where we put

$$2\Gamma=(p_3^{\,2}+q_3^{\,2})(1+e^2+x),$$

with $e^2=p_2^{\,2}+q_2^{\,2}$.

These changes are not necessary in the solution of the equation for τ, provided, in forming $x\partial R/\partial a$ we remember that a is present

in R through e, Γ when we transform to the canonical elements. The coefficient of x in the expansion given above can be obtained with sufficient accuracy from

$$\left(-2a\frac{\partial}{\partial a}-\tfrac{1}{2}e\frac{\partial}{\partial e}-\Gamma\frac{\partial}{\partial \Gamma}\right)\frac{a'R}{\mu},$$

and in this expression, R can be limited to the terms of the second order.

9·13. *The equation for* τ.

It has been shown in 9·7 that, by a suitable change of variables, the short period terms can be eliminated from the disturbing function, and that the equations for the new variables, which are distinguished by the suffix zero, have the same form as those for the old variables if we simply omit from the latter the short period terms. The suffix is unnecessary during the solution of the equations for the new variables and will be omitted.

The Hamiltonian function, namely,

$$\tfrac{3}{2}x^2-2x^3+\frac{a'}{\mu}\mathrm{R}_c+2x\frac{a'^2}{\mu}\frac{\partial \mathrm{R}_c}{\partial a},$$

will be denoted by

$$\tfrac{3}{2}x^2-2x^3+U+2xV, \quad \text{to order } m^{\frac{3}{2}}.$$

According to 9·7, with its reference to 9·5, U, V are independent of x, t.

The equations for x, τ are then

$$\frac{dx}{n'dt}=\frac{\partial U}{\partial \tau}+2x\frac{\partial V}{\partial \tau}, \quad \text{to order } m^{\frac{3}{2}}, \qquad \text{.............(1)}$$

$$\frac{d\tau}{n'dt}=-3x+6x^2-2V, \quad \text{to order } m. \qquad \text{......(2)}$$

Differentiate the last equation with respect to t. Since the derivatives of x, x_2, x_3, w_2, w_3 have the factor m and that of τ has the factor $m^{\frac{1}{2}}$, we obtain

$$\frac{d^2\tau}{n'^2dt^2}=(-3+12x)\frac{dx}{n'dt}-2\frac{\partial V}{\partial \tau}\frac{d\tau}{n'dt}, \quad \text{to order } m^{\frac{3}{2}}. \text{...(3)}$$

On substituting for dx/dt, $d\tau/dt$ from (1), (2), we see that the portions depending on V disappear and that the equation becomes

$$\frac{d^2\tau}{n'^2 dt^2} = -3(1-4x)\frac{\partial U}{\partial \tau}, \quad \text{to order } m^{\frac{3}{2}}. \quad \text{.........(4)}$$

The disappearance of V from the equation for τ, the solution of which is the principal part of the problem, is fortunate because it enables us to find τ with high accuracy without transforming the disturbing function to canonical variables. It is true that V is needed in the determination of x, but this is a comparatively simple problem when τ has been obtained in terms of the time.

Since $$x = -\frac{1}{3}\frac{1}{n'}\frac{d\tau}{dt}, \quad \text{to order } m^{\frac{1}{2}},$$

the transference of the factor $1-4x$ to the opposite side of (4), where it becomes a factor $1+4x$, gives the somewhat more convenient form of (4):

$$\frac{1}{n'^2}\frac{d^2\tau}{dt^2} - \frac{2}{3}\frac{d}{n'dt}\left(\frac{1}{n'}\frac{d\tau}{dt}\right)^2 = -3\frac{\partial U}{\partial \tau}, \quad \text{to order } m^{\frac{3}{2}}.$$

Finally, if we put

$$t_1 = t + \tfrac{4}{9}(\tau + \text{const.}),$$

it is easily seen that the equation reduces to

$$\frac{d^2\tau}{n'^2 dt_1^2} + 3\frac{\partial U}{\partial \tau} = 0, \quad \text{to order } m^{\frac{3}{2}}, \quad \text{............(5)}$$

which is the fundamental equation for the determination of the libration.

9·14. *First approximation to* τ.

We shall obtain this approximation on the assumption that x_2, x_3, w_2, w_3 are constants: it will appear later that their variable parts are divisible by $m^{\frac{1}{2}}$. With this assumption, U becomes a function of a single variable τ and the equation 9·13 (5) admits the integral

$$\frac{1}{n'^2}\left(\frac{d\tau}{dt_1}\right)^2 = -6U + \text{const.} = C - 6U,$$

so that $$n' t_1 + \text{const.} = \int (C - 6U)^{-\frac{1}{2}} d\tau$$

gives the solution. A reversion will give τ in terms of t_1.

From a theoretical point of view, this is sufficient to determine τ, but the process is inconvenient for calculation, and it is better to find τ as follows.

Since we know that τ oscillates about one of the values $\pm 60°$, let us put

$$\tau = \pm 60° + \delta\tau,$$

and expand the second term of 9·13 (5) in powers of $\delta\tau$. If U_i be the ith derivative of U with respect to τ with $\tau = \pm 60°$ inserted, the equation becomes

$$\frac{1}{n'^2} \frac{d^2}{dt_1^2} \delta\tau + 3U_1 + 3U_2 \delta\tau + \tfrac{3}{2} U_3 (\delta\tau)^2 + \tfrac{1}{2} U_4 (\delta\tau)^3 + \ldots = 0. \quad \ldots\ldots(1)$$

When $e = e' = \Gamma = 0$, the formulae of 9·3 give

$$U_1 = 0, \quad U_2 = \tfrac{9}{4} m, \quad U_3 = -\tfrac{27}{8} \sqrt{3} m, \quad U_4 = \tfrac{207}{16} m, \ldots.$$

In this case, when powers of $\delta\tau$ beyond the first are neglected, (1) reduces to

$$\frac{1}{n'^2} \frac{d^2}{dt_1^2} \delta\tau + \tfrac{27}{4} m \, \delta\tau = 0,$$

the solution of which is

$$\delta\tau = b \cos (\nu n' t_1 + \nu_0) = b \cos \phi, \quad \nu^2 = \tfrac{27}{4} m, \quad \ldots\ldots(2)$$

where b, ν_0 are arbitrary constants.

The result agrees with that obtained in 9·10. In the further approximations, we are assuming that b is a parameter which is small enough for expansions in powers of b to be possible. The largest known value for b is that in the case of Hector for which it is near ·3. As $\nu = {·079}$, we have $\frac{1}{3} \nu b = {·008}$ in this case. The statement in 9·5 that x is less than ·01 in all known (see footnote to 9·5) cases is thus justified.

In general, when constant values of e, e', Γ are used, we have, for the first approximation,

$$\delta\tau = b \cos (\nu n' t + \nu_0) = b \cos \phi, \quad \nu^2 = 3U_2. \quad \ldots\ldots(2')$$

For the next approximation, the value (2′) is substituted for $\delta\tau$ in the term $\frac{3}{2}U_3(\delta\tau)^2$ of (1) and the term $3U_1$ is included. The equation becomes

$$\frac{1}{n'^2}\frac{d^2}{dt_1^2}\delta\tau + 3U_2\delta\tau = -3U_1 - \tfrac{3}{4}U_3 b^2 - \tfrac{3}{4}U_3 b^2 \cos 2\phi. \quad \ldots(3)$$

The addition to $\delta\tau$ is the particular integral corresponding to the right-hand member. It is

$$-\frac{U_1}{U_2} - \frac{1}{4}\frac{U_3}{U_2}b^2 + \frac{1}{4}\frac{U_3}{U_2}b^2\frac{\cos 2\phi}{4-1}. \quad \ldots\ldots\ldots\ldots(4)$$

For the next approximation, we substitute the sum of (2), (4) in the term $\frac{3}{2}U_3(\delta\tau)^2$, and (2) in the term $\frac{1}{2}U_4(\delta\tau)^3$. The additional terms on the right of (3) are

$$-3U_3 b\cos\phi\left(-\frac{U_1}{U_2} - \frac{1}{4}\frac{U_3}{U_2}b^2 + \frac{1}{12}\frac{U_3}{U_2}b^2\cos 2\phi\right) - \tfrac{1}{2}U_4 b^3\cos^3\phi,$$

or
$$-3U_2 b\cos\phi\left(-\frac{U_1U_3}{U_2^2} - \frac{5}{24}\frac{U_3^2}{U_2^2}b^2 + \frac{1}{8}\frac{U_4}{U_2}b^2\right)$$
$$-\cos 3\phi\left(\frac{1}{8}\frac{U_3^2}{U_2}b^3 + \tfrac{1}{8}U_4 b^3\right).$$

The particular integral corresponding to the term with argument 3ϕ is

$$\frac{1}{3U_2}\cdot\frac{1}{9-1}\frac{b^3}{8}\left(\frac{U_3^2}{U_2} + U_4\right) = \frac{b^3}{192}\left(\frac{U_3^2}{U_2^2} + \frac{U_4}{U_2}\right).$$

In the term with argument ϕ we can put $b\cos\phi = \delta\tau$, since it has b^3 as a factor. On combining it with the first approximation we have

$$\frac{d^2}{n'^2 dt_1^2}\delta\tau + 3U_2\delta\tau\left(1 - \frac{U_1U_3}{U_2^2} - \frac{5}{24}\frac{U_3^2}{U_2^2}b^2 + \frac{1}{8}\frac{U_4}{U_2}b^2\right) = 0.$$

This shows that instead of the value $3U_2$ for ν^2, previously used, we must put

$$\nu^2 = 3U_2\left(1 - \frac{U_1U_3}{U_2^2} - \frac{5}{24}\frac{U_3^2}{U_2^2}b^2 + \frac{1}{8}\frac{U_4}{U_2}b^2\right),$$

in order that $\delta\tau$ may still remain periodic.

The process is continued as far as may be necessary. Each alternate approximation requires an addition to the value of ν.

The process of approximation followed above is that which is usual in the case of equations of the type (1). It is to be remembered that the value $3U_2$ is merely an approximation to ν^2 and that the latter must contain other terms if $\delta\tau$ is to remain periodic.

The equation may be solved also by putting

$$\delta\tau = b_0 + b\cos\phi + b_2\cos 2\phi + b_3\cos 3\phi + \dots$$

in (1) and equating to zero the coefficients of $\cos i\phi$. A series of equations of condition are obtained which have to be solved by continued approximation on the basis that b, b_0 and b_2, b_3, ... are quantities of the first, second, third, ... orders. The coefficient of $\cos\phi$ determines ν, and b is an arbitrary constant.

The arbitrary constants b, ν_0 replace the usual arbitrary constants n, ϵ which become fixed as soon as the triangular solution is adopted.

9·15. *The equations for the remaining variables.*

The development of the disturbing function in 9·8 shows that the long period terms containing the argument θ are of the fourth order. If we neglect terms of this order, the canonical equations give $dx_3/dt = 0$, or $x_3 = \text{const.}$ Further, since

$$x_3 = \Gamma(1-e^2)^{\frac{1}{2}} c_1/c_1' = \Gamma(1-e^2)^{\frac{1}{2}}(1+x),$$

we have, to the same degree of accuracy, $\Gamma = x_3 = \text{const.}$

The equation for θ is found with the aid of 9·12 (2) and is integrated after this expression has been developed as a Fourier series with argument ϕ with the solution given in 9·14.

To the same degree of accuracy, 9·12 (1) shows that $x_2 = -\frac{1}{2}e^2$.

The variables p_2, q_2, defined in 9·12 (2), reduce to

$$p_2 = e\sin\varpi, \quad q_2 = e\cos\varpi. \quad \dots\dots\dots\dots\dots(1)$$

The limitation enables us to neglect the part of R which has x as a factor. Hence, with the Hamiltonian function in 9·13, the equations for p_2, q_2 become

$$\frac{1}{n'}\frac{ds}{dt} = \frac{\partial U}{\partial c}, \quad \frac{1}{n'}\frac{dc}{dt} = -\frac{\partial U}{\partial s}, \quad \dots\dots\dots\dots\dots(2)$$

with the briefer notation s for p_2 and c for q_2.

The development of $a'R/\mu$, which contains c, s to the second order, is given by 9·9 (1). With $g-g'=\tau-\varpi+\varpi'$, the substitution (1), and the notation

$$T=\frac{\partial R_0}{\partial\tau},\quad P=2\frac{\partial^2 R_0}{\partial\tau^2}+\tfrac{1}{2}\alpha^2\frac{\partial^2 R_0}{\partial\alpha^2},$$

this development may be written

$$T\{-\mathrm{s}e'\cos(\tau+\varpi')+\mathrm{c}e'\sin(\tau+\varpi')\}$$
$$+\tfrac{1}{2}P\{\mathrm{s}^2+\mathrm{c}^2+e'^2-2e'\mathrm{s}\sin(\tau+\varpi')-2e'\mathrm{c}\cos(\tau+\varpi')\}.$$

The equations for s, c are therefore

$$\left.\begin{aligned}\frac{1}{n'}\frac{d\mathrm{s}}{dt}&=Te'\sin(\tau+\varpi')-Pe'\cos(\tau+\varpi')+P\mathrm{c},\\ \frac{1}{n'}\frac{d\mathrm{c}}{dt}&=Te'\cos(\tau+\varpi')+Pe'\sin(\tau+\varpi')-P\mathrm{s}.\end{aligned}\right\}\quad\ldots(3)$$

The only variable present in T, P is τ and we suppose that τ has been expressed as a function of t by means of the solution obtained in 9·14. Further, as $\tau=\pm 60^\circ+\delta\tau$, it is supposed that any of the functions of τ present can be expanded into Fourier series with argument ϕ.

9·16. *Solution of the equations for* s, c.

As is usual with linear equations, we first find the complementary function, which is the solution of

$$\frac{1}{n'}\frac{d\mathrm{s}}{dt}-P\mathrm{c}=0,\quad \frac{1}{n'}\frac{d\mathrm{c}}{dt}+P\mathrm{s}=0,\quad\ldots\ldots\ldots\ldots(1)$$

where $$P=P_0+P_1\cos\phi+P_2\cos 2\phi+\ldots,$$

with $\phi=\nu n't+\nu_0$; P has the factor m.

It is at once seen that the solution is

$$\mathrm{s}=e_0\sin(\varpi_1+\varpi_0),\quad \mathrm{c}=e_0\cos(\varpi_1+\varpi_0),$$

where $d\varpi_1/dt=P$, so that

$$\varpi_1=P_0n't+n'\Sigma\frac{P_i}{i\nu}\sin i\phi,\quad\ldots\ldots\ldots\ldots\ldots(2)$$

where e_0, ϖ_0 are arbitrary constants. We thus have

$$\left.\begin{aligned}\mathrm{s}&=e_0\sin\left(P_0n't+\varpi_0+\Sigma\frac{P_i}{i\nu}\cos i\phi\right),\\ \mathrm{c}&=e_0\cos\left(P_0n't+\varpi_0+\Sigma\frac{P_i}{i\nu}\cos i\phi\right).\end{aligned}\right\}\quad\ldots\ldots\ldots(3)$$

In the usual language of celestial mechanics $P_0 n'$ is the mean motion of the perihelion. It appears then that this part of the motion gives a constant 'eccentricity' and a variable motion to the perihelion. Since the coefficients $P_i/i\nu$ contain the factor $bm^{\frac{1}{2}}$, the solution to this order can be expressed in the form

$$\begin{aligned} \mathrm{s} &= e_0 \sin (P_0 n't + \varpi_0) \\ &\quad + \tfrac{1}{2}\Sigma \frac{P_i}{i\nu} \{\sin (P_0 n't + \varpi_0 + i\phi) - \sin (P_0 n't + \varpi_0 - i\phi)\}, \end{aligned}$$

with a similar form for c. The mean motion of the perihelion is divisible by m, the periodic part being divisible by $bm^{\frac{1}{2}}$.

The particular integral corresponding to the terms factored by e' in 9·15 (3) is required. These portions are functions of τ only and can therefore be expressed by Fourier series with argument ϕ. Suppose the solution to be of the form

$$\left.\begin{aligned} \mathrm{s} &= e' \Sigma\, (s_i \cos i\phi + s_i' \sin i\phi) = e'\,(\mathrm{s}_c + \mathrm{s}_s), \\ \mathrm{c} &= e' \Sigma\, (c_i \cos i\phi + c_i' \sin i\phi) = e'\,(\mathrm{c}_c + \mathrm{c}_s), \end{aligned}\right\} \quad \ldots\ldots(4)$$

so that the suffixes c, s denote expansions in cosines and sines of multiples of ϕ respectively. These are to be substituted in 9·15 (3). The coefficients of $\cos i\phi$, $\sin i\phi$ equated to zero will give the coefficients s_i, s_i', c_i, c_i'.

In a first approximation we retain only the terms of lowest order with respect to m in τ. According to 9·13, t_1 is then equal to t, and $\tau = \pm 60° + \Sigma b_i \cos i\phi$. The terms factored by e' in 9·15 (3) are then expressible by cosines of multiples of ϕ. The only sines which will be present will be those arising from $d\mathrm{s}_c/dt$, $d\mathrm{c}_c/dt$, s_s, c_s. We must therefore have

$$\frac{d\mathrm{s}_c}{dt} = P\mathrm{c}_s, \qquad \frac{d\mathrm{s}_s}{dt} = P\{\mathrm{c}_c - \cos(\tau + \varpi')\} + T \sin(\tau + \varpi'), \quad \ldots(5)$$

$$\frac{d\mathrm{c}_c}{dt} = -P\mathrm{s}_s, \quad \frac{d\mathrm{c}_s}{dt} = -P\{\mathrm{s}_c - \sin(\tau + \varpi')\} + T\cos(\tau + \varpi'). \ldots(6)$$

Since $d\phi/dt$ has the factor $m^{\frac{1}{2}}$, the required conditions can be satisfied only if

$$\begin{aligned} \mathrm{s}_c,\ \mathrm{c}_c &= \text{const.} + \text{cosines of multiples of } \phi, \text{ factor } m, \\ \mathrm{s}_s,\ \mathrm{c}_s &= \qquad\qquad\quad \text{sines of multiples of } \phi, \text{ factor } m^{\frac{1}{2}}. \end{aligned}$$

If, then, P_0 be the constant term in the expansion of P as a Fourier series with argument ϕ, the equations (5) and (6), with the notation (4), give

$$P_0 \mathrm{c}_0 = \text{const. term in the expansion of}$$
$$P\cos(\tau+\varpi') - T\sin(\tau+\varpi'),$$
$$P_0 \mathrm{s}_0 = \text{const. term in the expansion of}$$
$$-P\sin(\tau+\varpi') - T\cos(\tau+\varpi'),$$

to the order m.

Since the coefficients c_i, c_i', s_i, s_i' have at least the factor $m^{\frac{1}{2}}$, the terms $P\mathrm{c}_s$, $-P\mathrm{s}_s$ in (5), (6) can be neglected in finding a first approximation to these coefficients. Hence s_s, c_s are given by

$$\mathrm{s}_s = \int\{-P\cos(\tau+\varpi') + T\sin(\tau+\varpi')\}\,dt,$$

$$\mathrm{c}_s = \int\{\ \ P\sin(\tau+\varpi') + T\cos(\tau+\varpi')\}\,dt,$$

and they will have the factor $m^{\frac{1}{2}}$. When these have been found, the first of (5), (6) give s_c, c_c, the variable parts of which have the factor m.

The same plan is followed when terms factored by $m^{\frac{3}{2}}$ are retained in 9·15 (3). The work is simplified by remembering that in these expansions the coefficients of cosines of multiples of ϕ have even powers of $m^{\frac{1}{2}}$ as factors, while those of sines have odd powers as factors. The latter also have b as a factor. It follows that the errors of c_0, s_0 as determined in the approximation just given, have the factor m, those of c_i, s_i the factor m^2, and those of c_i', s_i' the factor $m^{\frac{3}{2}}$.

When $b=0$ we have $\tau = \pm 60°$, $T=0$, and P is reduced to a constant. The particular integral is then

$$\mathrm{s} = e'\sin(\pm 60° + \varpi'), \quad \mathrm{c} = e'\cos(\pm 60° + \varpi'), \quad \ldots(7)$$

and the complete solution is obtained by adding these to the complementary function which is

$$\mathrm{s} = e_0\sin(P_0 t + \varpi_0), \quad \mathrm{c} = e_0\cos(P_0 t + \varpi_0). \quad \ldots\ldots(8)$$

Since the mean longitudes of the asteroid and Jupiter differ only by the constant $\pm 60°$, the terms (7) can be interpreted by the statement that Jupiter impresses its elliptic terms on the motion of the asteroid. This result might have been anticipated from the triangular solution given in 9·2, for the mass of Jupiter is so large compared with that of the asteroid that in this solution its elliptic motion will be dominant.

In the ordinary planetary theory, the terms (8) would be expanded in powers of t. Thus, neglecting terms factored by m^2, we have

$$\mathrm{s} = e_0 \sin \varpi_0 + P_0 t e_0 \cos \varpi_0, \quad \mathrm{c} = e_0 \cos \varpi_0 - P_0 t e_0 \sin \varpi_0.$$

The constants (7) might therefore have been supposed to be included in the arbitraries $e_0 \sin \varpi_0$, $e_0 \cos \varpi_0$, which are to be determined from observation. Thus although these constants are affected by an error of order m in the first approximation, most of the error will be absorbed in the determination of e_0, ϖ_0 from observation. Owing to the need for further approximation, however, they must be kept separated.

The fact that s, c differ from constants by terms having the factor $m^{\frac{1}{2}}$ at least, justifies the assumption made in 9·14 that e, ϖ may be treated as constants in finding a first approximation to τ.

9·17. *Higher approximations and final results.*

The equations for τ, x have been taken to the order $m^{\frac{3}{2}}$ and solved with s, c, p_3, q_3 constant. Since the variable parts of the latter have the factor $bm^{\frac{1}{2}}$, it follows that the errors of the equations for τ, x have the factor $bm^{\frac{3}{2}}$ and also a factor of the order of the squares of the eccentricities and inclination. The error of τ has therefore this latter factor and also the factor $bm^{\frac{1}{2}}$.

For the next approximation, the variable values of s, c, p_3, q_3 are inserted in U. Since the resulting addition to τ will be small, it will in general be sufficient to find it from the following equation, deduced from 9·13 (5),

$$\left(\frac{1}{n'^2}\frac{d^2}{dt^2} + \nu^2\right)\delta\tau = \text{additional terms in } -\frac{\partial U}{\partial \tau};$$

the additional part of x can then be obtained. If further approximations to the values of the remaining variables are needed, they can be obtained in a similar manner.

The final results give the values of τ_0, x_0, x_{20}, x_{30}, w_{20}, w_{30}, or of the variables which replace them, where the suffix zero which was dropped according to the statement in the first paragraph of 9·13 is now replaced. The values of the original variables are then found by substituting these results in 9·7 (2). Now the portions dependent on the derivatives of S in these latter equations all have the factor m, so that the effect of substituting variable for constant values of the variables with suffix zero will be very small, with one exception, that of τ now called τ_0, and for the latter the portion independent of m will serve.

9·18. *Numerical developments.*

A literal theory in which the expansions are made in powers of b, e_0, Γ_0, e', $m^{\frac{1}{2}}$ can be formed which will give a close approximation without an excessive amount of calculation. Even with so large a value of b as ·3, the series for τ converges rapidly owing to the numerical divisors which the integration produces. In this respect the theory of the Trojan group differs from the ordinary planetary theory where expansions in powers of α converge so slowly that they are useless for numerical calculation.

The work can, however, be greatly simplified when the numerical values of the parameters are known. The chief part of τ is a Fourier series with argument ϕ and most of the further calculations consist in the calculation of various functions of $\cos\tau$, $\sin\tau$. The functions are rapidly calculated if harmonic analysis be used; five, or at most seven, special values of ϕ will be sufficient. Analyses of the special values of the functions are needed only when they have to be integrated or differentiated.

Harmonic analysis can also be used conveniently to complete the solution of 9·13 (5) when an approximate value of τ has been obtained by the method of 9·14. Suppose that such an approximate value is

$$\tau = \pm 60^\circ + \Sigma b_i \cos i\phi, \qquad d\phi/dt = \nu, \quad \text{..................(1)}$$

and let the required correction be

$$\delta\tau = \Sigma \delta b_i \cos i\phi - t\,\delta\nu\, \Sigma b_i\, i \sin i\phi, \qquad \delta b_1 = 0, \quad \text{...........(2)}$$

where we neglect squares of the correction to ν. As the arbitrary constant can be left unchanged, we can put $\delta b_1 = \delta b = 0$. With the value (1) of τ, the function $\partial U/\partial\tau$ is computed by harmonic analysis and compared with $-d^2\tau/n'^2 dt^2$. Let the sum of these be denoted by

$$\epsilon_0 + \epsilon_1 \cos\phi + \epsilon_2 \cos^2\phi + \dots.$$

Then a further approximation will be obtained by solving the equation

$$\frac{d^2}{n'^2 dt^2}\delta\tau + 3\frac{\partial^2 U}{\partial\tau^2}\delta\tau = -\epsilon_0 - \epsilon_1\cos\phi - \ldots,$$

in which $\partial^2 U/\partial\tau^2$ is computed like $\partial U/\partial\tau$.

On substitution of the expression (2) for $\delta\tau$, it will be found that the coefficient of t disappears and that the values of δb_i, $\delta\nu$ can be obtained by equating to zero the coefficients of $\cos i\phi$. The process can be repeated if necessary. Since the principal part of $3\partial^2 U/\partial\tau^2$ is ν^2, the principal part of $\delta\tau$ is found at once.

9·19. *Determination of the constants from observation.*

The nature of the orbit is usually set forth by giving the values of the osculating elements at some given date: these are found from the observations by methods which are outside the scope of this volume. A procedure for finding the values of the constants used in the theory from these elements is contained in the following plan.

An approximation to the short period terms can be obtained by substituting for the elements with suffix zero in the terms arising from S, their osculating values. The same procedure is followed with the terms due to the action of Saturn determined below. We then obtain an approximation to the elements with suffix zero by the use of the equations which connect them with the actual osculating elements.

By comparison of the elements with suffix zero with the literal series for τ, x in powers of b, values of b, ν_0 are obtained. With these the short period terms, particularly those dependent on the angle ϕ, can be calculated again and the same procedure repeated. At this stage the values of the constants attached to the remaining elements can be found with high accuracy. If necessary, the whole procedure may be repeated, but it will rarely be necessary to do so except perhaps for the constants b, ϕ_0, which are sensitive to small changes in the elements.

The process does not differ essentially from that which would be followed in the ordinary planetary theory if the methods of Chap. VI be used to determine the perturbations. In the latter, however, the elements with suffix zero contain the long period and secular terms only, and if desired we can treat these like the short period terms, using the observed osculating elements to find a first approximation to n_0, ϵ_0, etc. Thus while the methods of Chap. VI have certain disadvantages which have been pointed out in 6·25, the custom of defining an orbit by giving the values of its osculating elements at a given date, makes the determination of the constants of the orbit from these values a simple problem.

In the case of the Trojan group, the following modification gives the constants more rapidly. The values of the osculating elements are found at several dates by carrying them forward or backward by the method of

special perturbations. Since all the short period terms have periods approximating to that of revolution round the sun or sub-multiples of this period, a mean value of an element with suffix zero can be found by analysing its values at the various dates into a Fourier series with argument $2\pi/n'$ and choosing the constant term as the first approximation to its value at a mean date.

PERTURBATIONS BY SATURN

9·20. The calculation of the perturbations of an asteroid of the Trojan group by Saturn is difficult because the procedure of the ordinary planetary theory cannot be followed. This procedure consists in finding the perturbations due to each planet separately, then those due to their combined actions, and adding the results. Here, in finding the principal perturbations produced by Saturn, we cannot neglect the action of Jupiter, even in a first approximation. Thus the problem is one of four bodies rather than of three, and in this respect it is similar to that of the action of the planets on the moon.

The disturbing function for the *direct* attraction of Saturn is that given by 1·10 (1). An *indirect* effect is also produced by the action of Saturn in causing Jupiter to deviate from elliptic motion, so that, in the disturbing function due to the direct action of Jupiter, it is necessary to add to the elliptic elements of that planet the perturbations caused by Saturn. An indirect effect of the action of Jupiter is also present in the perturbations this planet produces on the motion of Saturn.

It is assumed that the mutual perturbations of Jupiter and Saturn on one another are completely known. The largest term in the action of Saturn on Jupiter has a coefficient in the longitude of Jupiter of nearly 1200″ and a period of some 870 years, this long period being due to the fact that the period of revolution of Saturn is nearly $2\frac{1}{2}$ times that of Jupiter. Since the period of revolution of the asteroid is the same as that of Jupiter, we might expect to find a term of similar magnitude in the motion of the asteroid. It will be shown, however, that the action of Jupiter fundamentally alters the direct effect of the action of Saturn, and that the indirect effect produced by Saturn is the largest part of the action of that planet.

It will be assumed that these effects are, in general, small compared with those which we have been considering in the first part of this chapter, and that, in developing the disturbing functions due to Saturn, we can put for the coordinates of the asteroid their values in terms of the time. These additional portions can then be separated into long and short period terms. The latter may be eliminated as before by a change of variables which will give additional portions to the function S. We have then to consider the effect of small additive terms in the function U on the variables with suffix zero; as before this suffix will be dropped until the final results for the long period terms have been obtained.

9·21. *The equation for $\delta\tau$ and its solution.*

The equation for τ given in 9·13, namely,

$$\frac{1}{n'^2}\frac{d^2\tau}{dt_1{}^2}+3\frac{\partial U}{\partial\tau}=0, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

is still true when we add to U the portions due to the actions of other planets, provided the conditions laid down remain satisfied. These conditions demanded that the principal part of x should be given by $x=-\frac{1}{3}d\tau/n'dt_1$, and that the terms present in x should be large compared with the corresponding terms in U.

With the conditions laid down in the last paragraph of 9·20, the inclusion of the action of Saturn will require the solution of equations of the form

$$\frac{1}{n'^2}\frac{d^2\tau}{dt_1{}^2}+3\frac{\partial U}{\partial\tau}=A\sin(n'pt+p_0), \quad \ldots\ldots\ldots\ldots(2)$$

where U has the meaning previously given and A, p, p_0 are known constants. To simplify the exposition, we shall put $t_1=t$; this amounts to the neglect of terms of order higher than those retained.

Let $\tau=\tau_0$ be the solution of (1) and $\tau=\tau_0+\delta\tau$ that of (2). If squares and higher powers of $\delta\tau$ be neglected, we have

$$\frac{1}{n'^2}\frac{d^2}{dt^2}\delta\tau+3\frac{\partial^2 U}{\partial\tau_0{}^2}\delta\tau=A\sin(pn't+p_0), \quad \ldots\ldots(3)$$

where in U and its derivatives the value $\tau=\tau_0$ is inserted.

The solution of (1) gave τ_0 as a function of t and of two arbitrary constants b, ν_0; if this solution be substituted in (1) the constants b, ν_0 disappear identically. We can therefore differentiate (1) with respect to b and ν_0 and obtain

$$\frac{1}{n'^2}\frac{d^2}{dt^2}\left(\frac{\partial\tau_0}{\partial b}\right)+3\frac{\partial^2 U}{\partial\tau_0{}^2}\cdot\frac{\partial\tau_0}{\partial b}=0,\quad \frac{1}{n'^2}\frac{d^2}{dt^2}\left(\frac{\partial\tau_0}{\partial\nu_0}\right)+3\frac{\partial^2 U}{\partial\tau_0{}^2}\cdot\frac{\partial\tau_0}{\partial\nu_0}=0. \quad\ldots\ldots(4)$$

Hence
$$\delta\tau=\frac{\partial\tau_0}{\partial b},\quad \delta\tau=\frac{\partial\tau_0}{\partial\nu_0}\quad\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

are particular solutions of (3) when $A=0$.

It follows from a well-known theorem that a particular solution of (3), corresponding to $A\neq 0$, is

$$\delta\tau=\frac{1}{C}\cdot\frac{\partial\tau_0}{\partial b}\int\frac{\partial\tau_0}{\partial\nu_0}A\sin(n'pt+p_0)\,dt$$
$$-\frac{1}{C}\cdot\frac{\partial\tau_0}{\partial\nu_0}\int\frac{\partial\tau_0}{\partial b}A\sin(n'pt+p_0)\,dt,\quad\ldots\ldots(6)$$

where
$$C=\frac{d}{dt}\left(\frac{\partial\tau_0}{\partial b}\right)\cdot\frac{\partial\tau_0}{\partial\nu_0}-\frac{d}{dt}\left(\frac{\partial\tau_0}{\partial\nu_0}\right)\cdot\frac{\partial\tau_0}{\partial b}=\text{const.}\quad\ldots\ldots(7)$$

Equation (6) may be tested by substitution in (3), and equation (7) by eliminating $\partial^2U/\partial\tau_0{}^2$ from (4).

Since τ_0 has been obtained as a Fourier series with argument $\phi=\nu n't+\nu_0$, the derivatives of τ_0 with respect to ν_0, b will still be Fourier series provided ν be independent of b. Actually, ν is a function of b and $\partial\tau/\partial b$ contributes a non-periodic portion $n't(\partial\nu/\partial b)(\partial\tau/\partial\nu_0)$; it is easily seen, however, that this non-periodic part disappears from (6).

The principal term in $\tau_0\mp 60°$ is $b\cos(\nu n't+\nu_0)$, where $\nu^2=27m/4$. With this value of τ_0 we obtain

$$\delta\tau=\frac{A}{\nu^2-p^2}\sin(n'pt+p_0),\quad\ldots\ldots\ldots\ldots\ldots(8)$$

a result which might be deduced directly from (3) since in this case $3\partial^2U/\partial\tau^2=\nu^2$. With the complete value of τ, the divisor ν^2-p^2 will be replaced by divisors of the form $i^2\nu^2-p^2$.

9·22. *Indirect perturbations.*

These arise through the substitution for the elements of Jupiter their disturbed instead of their elliptic values. We shall suppose that the perturbations of the plane of Jupiter's orbit can be neglected so that we can still use it as a fixed plane of reference. The perturbations to be considered are therefore those of a', e', $w_1' = w'$, $w_2' = \varpi'$.

These perturbations, chiefly due to Saturn, have the mass of Saturn as a factor and are substituted in an expression which has the mass of Jupiter as a factor. The resulting short period terms will be very small and, in any case, they are supposed to have been eliminated by the change of variables. It is found that the only terms likely to produce sensible effects are those of long period in the motion of Jupiter, producing terms of long period in the motion of the asteroid.

For their calculation, we return to the point in the original canonical equations where the transformation $\tau = w - w'$ was made (9·5). This transformation still holds if w' is any function of t independent of the elements of the asteroid provided that the expression $c_1 dw'/dt$ be added to the Hamiltonian function. As before, w' is then no longer present explicitly in the disturbing function.

If we denote by $\delta a'$, $\delta e'$, $\delta\varpi'$ the perturbations of a', e', ϖ', the equations for c_1, τ become

$$\frac{dc_1}{dt} = \frac{\partial R}{\partial \tau} + \frac{\partial^2 R}{\partial \tau \partial a'}\delta a' + \frac{\partial^2 R}{\partial \tau \partial e'}\delta e' + \frac{\partial^2 R}{\partial \tau \partial \varpi'}\delta\varpi', \qquad \text{(1)}$$

$$\frac{d\tau}{dt} = \left(\frac{\mu^2}{c_1^3} - n_0' - \frac{d}{dt}\delta w'\right) - \frac{\partial R}{\partial c_1} - \frac{\partial^2 R}{\partial c_1 \partial a'}\delta a' - \frac{\partial^2 R}{\partial c_1 \partial e'}\delta e' - \frac{\partial^2 R}{\partial c_1 \partial \varpi'}\delta\varpi', \qquad \text{(2)}$$

where n_0' is the constant term in dw'/dt.

We have seen in Chap. VI that the variations $\delta a'$, $\delta e'$, $\delta\varpi'$ contain the first powers only of the small divisor which is present in the case of a long period term, and these variations, when multiplied by the second derivatives of R, will give very small terms which can be neglected, at least in a first approximation.

If the procedure followed in 9·6 and 9·11 be then adopted, it is easily seen to lead to the equation

$$\frac{1}{n'^2}\frac{d^2\tau}{dt_1^2}+3\frac{\partial U}{\partial \tau}=-\frac{1}{n'^2}\frac{d^2}{dt^2}(\delta w'), \quad \ldots\ldots\ldots\ldots(3)$$

an equation which replaces 9·13 (5). To find the principal term we put $t_1 = t$.

Suppose $$\delta w' = B\sin(n'pt + p_0),$$

and that we substitute this in (3). According to 9·21 (8) the principal part of $\delta\tau$ is given by

$$\delta\tau = \frac{Bp^2}{\nu^2 - p^2}\sin(n'pt + p_0).$$

But $\delta w = \delta\tau + \delta w'$. Hence

$$\delta w = \frac{B\nu^2}{\nu^2 - p^2}\sin(n'pt + p_0). \quad \ldots\ldots\ldots\ldots(4)$$

The effect of a perturbation in w' therefore depends on the relative magnitudes of ν, p.

If ν is large compared with p, (4) gives approximately

$$\delta w = B\sin(n'pt + p_0) = \delta w'.$$

The result $\delta w = \delta w'$ applies also to a secular term since such a term can be expressed as one of very long period. Hence the general proposition:

If the period of a perturbation of Jupiter by Saturn or by any other planet is long compared with the period of libration of the asteroid (about 150 years), the perturbation of the longitude of Jupiter is directly impressed on the longitude of the asteroid.

The principal perturbation of Jupiter by Saturn has a period of some 870 years, so that the indirect perturbation of the asteroid differs from the direct perturbation in the motion of Jupiter by less than three per cent., although this indirect effect is one of the second order with respect to the masses while the direct effect on Jupiter is one of the first order.

Terms in which p is large compared with ν have been treated as short period effects and therefore do not enter into the discussion.

Terms in which p^2 is nearly equal to ν^2 would give rise to much larger perturbations in the motion of the asteroid than those present in the motion of Jupiter. There are no such terms having sensible coefficients in the perturbations produced by Saturn. Neptune, which has a period of 164 years, comes nearest to producing such a term.

9·23. To obtain the perturbations of x, the equation 9·13 (1) is used. When p/ν is small, $\delta\tau$ is small compared with $\delta w'$ and therefore $(d/dt)\,\delta\tau$ with $(d/dt)\,\delta w'$. The latter is, however, large compared with $\partial R/\partial c_1$ which has the factor m and has no small divisor. Hence, since $\mu^2/c_1{}^3 = (1+x)^{-3}\,n'$,

$$-3\delta x - \frac{1}{n'}\frac{d}{dt}(\delta w') = 0,$$

or, since $\delta x = \frac{1}{2}\delta a/a'$, this gives $(d/dt)\,\delta w' = -\frac{3}{2}\delta n'$. Hence

$$\delta a = \delta a'.$$

The long period terms in the major axis of Jupiter are therefore impressed on that of the asteroid.

For the perturbations of e, ϖ, the equations 9·15 (3), which are still true if e', ϖ' are variable, are used. Let e', ϖ' receive long period variations $\delta e'$, $\delta\varpi'$. Then, as before, it may be shown that $P\mathrm{c}$, $P\mathrm{s}$ may be neglected. When the libration vanishes, the equations for $\delta\mathrm{s}$, $\delta\mathrm{c}$ reduce to

$$\frac{d}{dt}\delta\mathrm{s} = -P_0\delta\,\{e'\cos(\pm 60^\circ + w')\},\quad \frac{d}{dt}\delta\mathrm{c} = P_0\delta\,\{e'\sin(\pm 60^\circ + w')\}.$$

If $2\pi/n'p$ be the period of the variations of $\delta e'$, $\delta\varpi'$, the corresponding coefficients in $\delta\mathrm{s}$, $\delta\mathrm{c}$ are diminished in the ratio P_0/p which is usually small, so that $\delta\mathrm{s}$, $\delta\mathrm{c}$ will be negligible. For the terms depending on ϕ when $b \neq 0$, the divisor is approximately $i\nu$ if p is small compared with ν and the coefficients of such terms in $\delta\mathrm{s}$, $\delta\mathrm{c}$ will be still smaller than those just treated. Thus the long period variations of e', ϖ' are not impressed on the asteroid but produce effects which are much smaller than those in the motion of Jupiter.

9·24. *The direct action of Saturn.*

The disturbing function for this action is given by 1·10 (1), and it can be expanded in terms of the elements of the asteroid and of Saturn by one of the methods used in the ordinary planetary theory. The expansion, if made in a literal form, is available whether the elements be constant or variable. The disturbing function, denoted by R', adds a term $a'R'/\mu$ to the Hamiltonian function in 9·5 (5).

The variable x which measures the deviation of c_1/c_1' from unity is still small, so that 9·5 (6) still holds when we add R' to R.

Let the elimination of the short period terms, as made in 9·7, include those arising from Saturn. For simplicity, we shall retain the lowest power of x only, so that the additional terms in S_0 are similar to those found in Chap. VI. The Hamiltonian function used in 9·13 therefore becomes

$$\tfrac{3}{2}x^2 + \frac{a'}{\mu}(R_c + R_c') = \tfrac{3}{2}x^2 + U + U',$$

with the notation adopted there. The equations 9·13 (1), (2) then become

$$\frac{1}{n'}\frac{dx}{dt} = \frac{\partial U}{\partial \tau} + \frac{\partial U'}{\partial \tau}, \quad \frac{1}{n'}\frac{d\tau}{dt} = -3x - 2V - \frac{\partial U'}{\partial x},$$

and equation 9·13 (3) reduces to

$$\frac{1}{n'^2}\frac{d^2\tau}{dt^2} = -3\frac{1}{n'}\frac{dx}{dt} - \frac{d}{dt}\left(\frac{\partial U'}{\partial x}\right).$$

On substituting for dx/dt, this gives

$$\frac{1}{n'^2}\frac{d^2\tau}{dt^2} = -3\left(\frac{\partial U}{\partial \tau} + \frac{\partial U'}{\partial \tau}\right) - \frac{d}{dt}\left(\frac{\partial U'}{\partial x}\right).$$

For a term of long period, the last term of this equation is small compared with $\partial U'/\partial \tau$, and may be neglected in a first approximation. The equation then reduces to

$$\frac{1}{n'^2}\frac{d^2\tau}{dt^2} + 3\frac{\partial U}{\partial \tau} = -3\frac{\partial U'}{\partial \tau}.$$

The right-hand member of this equation has the mass of Saturn as a factor, and we can substitute for the elements of the

asteroid the values obtained from the action of Jupiter alone. In accordance with the previous work, it will be sufficiently accurate to use the series for τ and to take all the other elements constant.

Let any term in the right-hand member be denoted by

$$A \sin(pn't + p_0).$$

The method of 9·21 is now available for the solution of the equation. The principal part of the addition to τ will be given by

$$\delta\tau = \frac{A}{\nu^2 - p^2} \sin(pn't + p_0).$$

The principal long period term due to the action of Saturn will have nearly the same period as it produces in the motion of Jupiter, namely, about 870 years, and therefore p is small compared with ν. Hence the small divisor due to a long period term is much larger than in the ordinary planetary theory and the resulting effect much smaller. Thus

Jupiter not only impresses on the asteroid its own inequalities of period long in comparison with that of the libration, but it also prevents the asteroid from having any very large terms of this nature arising from the action of another planet.

The substitution of the series for τ will also produce terms with arguments $(p \pm i\nu)n't + p_0 \pm i\nu_0$. When $i = 1$ we shall have divisors $\nu^2 - (p \pm \nu)^2$ or $\pm 2p\nu$ approximately. Such terms will have the factor b, and if b is large they may be sensible.

The theory of the perturbations of Jupiter by Saturn given by Leverrier* can be utilised for calculating the perturbations of the asteroid by Saturn, since the numerical value of the ratio of the mean distances is the same, and since Leverrier gives the contribution of each separate power of the eccentricities and inclination, so that the change to those of the asteroid can be easily computed. But the convergence along powers of the inclination is so slow when the inclination is large, as in the cases of certain members of the group, that the value of the coefficient of the principal long period term obtained in this way is doubtful. Another difficulty arises from the fact that the mean motion of the perihelion of the asteroid is comparable with that of the argument of this term so that it cannot be

* *Paris Obs. Mém.* vol. x.

neglected, even in a first approximation: see "Theory of the Trojan Group" referred to in 9·25.

9·25. The literature connected with the triangular solutions of the problem of three bodies is extensive: much of it is concerned with the possible orbits which may be described under different conditions but which have no present applications in the solar system. Amongst the earlier developments arriving at a more general theory for the asteroids of the Trojan group may be mentioned those of L. J. Linders (*Stockholm Vet. Ak. Arkiv*, Bd. 4, No. 20) and W. M. Smart (*Mem. Roy. Astron. Soc.* vol. 62, pp. 79–112, 1918); the latter used the method adopted by Delaunay for the development of the lunar theory. In a paper by E. W. Brown (*Mon. Not. Roy. Astron. Soc.* vol. 71 (1911), pp. 438–454), the particular periodic solution which constitutes the principal part of the libration is shown to be a linkage between the orbits of planets outside and inside that of Jupiter and those of satellites of Jupiter, the passages between them going through the collinear solutions. The linkage bears some resemblance to that which joins the two sets of solutions of the equation for the motion of a pendulum.

A literal development sufficiently complete to give the position of an asteroid of the group within a few seconds of arc has been made by E. W. Brown ("Theory of the Trojan Group of Asteroids," *Trans. of Yale Obs.* vol. 3, pp. 1–47, 87–133). The method of Chap. VII is used for the development of the action of Jupiter: the various problems which arise in finding this action are closely analogous to those set forth in this chapter. The problems and theorems connected with the action of Saturn are dealt with in detail. The theory was applied to the asteroid Achilles. This numerical application has been revised by D. Brouwer (*Trans. of Yale Obs.* vol. 6, pt. VII) who has added tables for finding its position at any time. W. J. Echert has applied the same theory to Hector (*Ib.* vol. 6, pt. VI), the libration of which runs up to over 20°.

A. APPENDIX ON NUMERICAL HARMONIC ANALYSIS

A·1. Let $F_c(x)$ be a periodic function of x, period 2π, which is expansible in the form

$$F_c(x) = c_0 + c_1 \cos x + c_2 \cos 2x + \dots + c_n \cos nx + \dots \quad \dots\dots(1)$$

The problem under consideration is the rapid numerical calculation of the coefficients c_i. It is supposed that $F_c(x)$ contains numerical constants only, and that after some term $c_n \cos nx$ the remainder of the series may be neglected.

Under the latter condition (1) may be regarded as an identity satisfied for all values of x. If then we choose $n+1$ numerical values of x and calculate the corresponding values of $F_c(x)$ and of the cosines of the multiples of x, we obtain $n+1$ relations which may be regarded as $n+1$ linear equations, giving the $n+1$ unknowns $c_0, c_1, \dots, c_n$.

The effectiveness of the method depends on the ease with which the special values of $F_c(x)$ can be computed and on that with which the linear equations may be solved. It is found that the questions principally to be considered are the choice of the special values of x and the best arrangement of the work for finding the c_i.

Similar remarks apply to the development of an odd function of x in the form

$$F_s(x) = s_1 \sin x + s_2 \sin 2x + \dots + s_n \sin nx + \dots, \quad \dots\dots(2)$$

except that n values of x only are needed since there is no constant term.

If $F(x)$ contains both even and odd functions of x and if we stop at the nth harmonic, $2n+1$ special values of x are needed

In cases where the calculation of special values of $F_c(x)$ and of $F_s(x)$ is needed and where much of the work is the same in each case, the same special values of x should be used.

A·2. *Choices of the special values of x.* If $F(x)$ contains both even and odd functions of x, they can be separated by choosing the values of x in pairs α, $2\pi-\alpha$. Since

$$\cos i\alpha = \cos i(2\pi-\alpha), \quad \sin i\alpha = -\sin i(2\pi-\alpha),$$

we have

$$F(\alpha)+F(2\pi-\alpha) = \text{cosine series}, \quad F(\alpha)-F(2\pi-\alpha) = \text{sine series}.$$

In future it will be supposed that this separation has been effected so that the forms (1) and (2) of A·1 can be considered separately.

Next, since

$$F_c(\alpha)+F_c(\pi-\alpha), \quad F_s(\alpha)-F_s(\pi-\alpha)$$

contain even multiples of x only, while

$$F_c(\alpha)-F_c(\pi-\alpha), \quad F_s(\alpha)+F_s(\pi-\alpha)$$

contain odd multiples only, the choice of pairs of supplementary values enables us to separate the equations giving the c_i or the s_i into two sets, one containing the coefficients with odd suffixes and the other those with even suffixes.

Finally, choices of α which are multiples of π/n, where n is an integer, have obvious advantages. The special cases $n=3$, 4 or 6 will suffice for most of the requirements in the plans for developing the disturbing function and the disturbing forces outlined here. For the rare cases in which eight harmonics are needed, it is advisable to add the values $x=45°$, $135°$ to those given by $n=6$, since the work done with six harmonics only can be fully utilised and repetition of it is not needed. Schedules for this last case are to be found in *Trans. of Yale Obs.* vol. 6, part 4, pp. 61–65.

A·3. *Determination of the nth harmonic in F_s.* Since F_s vanishes for $x=0$, π, these values are useless for the computation of the s_i. Moreover, since $\sin nx$ vanishes when x is any multiple of π/n, it is the coefficient of the nth harmonic which is undetermined with this set of values of x. It is sometimes possible to estimate its coefficient with sufficient accuracy; where this is not the case, one of the following devices may be adopted.

For $n=3$, the chosen values of x are 0°, 60°, 120°, 180°. The value $x=90°$ may be added to find s_3. If this value be used also in F_c, we can find c_4 and then get higher accuracy for c_2 (see A·4).

For $n=4$, 6 we may proceed as follows: In most cases the calculation of dF_s/dx for $x=0, \pi$ will be found to be easy. These give the values of $s_1 \pm 2s_2 + 3s_3 \pm 4s_4 + \ldots$ which, combined with the relation furnished by the other values of x inserted in F_s, will give the value of s_n. Only one of the two values is necessary, but the solution of the equations is simplified by using both: they give also the $(n+1)$th harmonic and higher accuracy to the $(n-1)$th harmonic.

A·4. *Errors of the coefficients.* When multiples of π/n are used for the special values of x, the solution of the linear equations for the cosine series actually gives, instead of c_i, $i<n$, the value of

$$c_i + c_{2n-i} + c_{2n+i} + c_{4n-i} + \ldots,$$

if we include all the terms of the series. Thus the principal part of the error of c_i is the rejected coefficient c_{2n-i}. When $i=n$, we actually determine the value of

$$c_n + c_{3n} + c_{5n} + \ldots,$$

so that the error of c_n is c_{3n}. Hence the lower the harmonic the higher the accuracy with which it is found, with the exception of the nth which has an error equal to the coefficient of the $3n$th harmonic, approximately.

In the case of F_s we determine the value of

$$s_i - s_{2n-i} + s_{2n+i} - s_{4n-i} + \ldots$$

instead of s_i, for $i<n$, so that the principal part of the error of s_i is $-s_{2n-i}$, and the same rule with respect to the errors of the coefficients of the lower harmonics holds.

When the harmonic with coefficient s_n (which vanishes when x is a multiple of π/n) is found for n odd by using the additional value $x=90°$, it is easily seen that we actually determine

$$s_n - 2s_{n+2} + \ldots;$$

for without its use we find $s_{n-2}-s_{n+2}+\ldots$, and with its use we find $s_{n-2}-s_n+s_{n+2}-\ldots$, so that the principal part of the error of s_n is $-2s_{n+2}$. If $x=90^\circ$ be also used in finding F_c, we determine c_{n-1} with an error c_{n+3} and c_{n+1} with an error $-c_{n+3}$, the errors of the other coefficients being unaffected.

When n is even, the use of the additional values $\partial F/\partial x$ with $x=0,\ \pi$ gives us

$$s_{n-1}+s_{n+3}+\ldots,\qquad s_n+2s_{n+2}+\ldots,\qquad s_{n+1}+s_{n+3}+\ldots,$$

so that the principal part of the error of s_{n-1} is s_{n+3} and that of s_n is $2s_{n+2}$, the remaining earlier harmonics being unaffected. Thus in the case of F_s the nth harmonic has the greatest error—exactly opposite to the case of F_c where it had the smallest error.

A·5. The schedules made out below are given in detail so that they may be used without preparation for harmonic analysis either for a single function or for many functions. In the latter case, the work should be carried out in parallel columns, any one step being the performance of the same operation in all the columns; since most of the operations are simple additions and subtractions, much time can be saved by this reduction to routine computation.

In the cases where the coefficients are known to diminish with some degree of regularity along values of i the results themselves furnish a test of the accuracy of the work; in any case, one or two of the special values may be reproduced from the final results with but little labour.

A·6. *Notation for the schedules.* The function to be analysed is denoted by F and the argument by x. For an even function, we put

$$F=c_0+c_1\cos x+c_2\cos 2x+\ldots,$$

and for an odd function

$$F=s_1\sin x+s_2\sin 2x+\ldots.$$

For the special values of x,

$$0^\circ,\ 180^\circ;\ 90^\circ;\ 45^\circ,\ 135^\circ;\ 30^\circ,\ 150^\circ;\ 60^\circ,\ 120^\circ;$$

the corresponding values of F will be denoted respectively by

$$F_0,\ F_0';\ F_9;\ F_4,\ F_4';\ F_3,\ F_3';\ F_6,\ F_6';$$

and, in the case of an odd function,

$$\left(\frac{dF}{dx}\right)_{x=0} \text{by } F_d, \quad -\left(\frac{dF}{dx}\right)_{x=\pi} \text{by } F_d'.$$

The letters $A, B, C, \ldots$ used in the schedules are defined therein

The first column of each schedule gives the symbols; the second column gives their numerical values in the examples the third, omitted in the computing forms, is explanatory. The error of any coefficient is shown in the final values. Thus $c_3\,(+\,c_9)$ indicates that if c_9 were known it would be subtracted in order to find c_3.

Since the same function has been used in all the examples for the cosine analyses and its derivative in the sine analyses, direct comparison of the errors of the various results is possible.

A·7. Schedule for 3 Harmonics. Cosine Series.

Values of $x=0°,\ 180°,\ 60°,\ 120°$. $F=(1-\cdot 6\cos x)^{\frac{1}{2}}$.

F_0	·63246	
F_0'	1·26491	
F_6	·83666	
F_6'	1·14018	
$A=F_0+F_0'$	1·89737	$2c_0+2c_2$
$B=F_6+F_6'$	1·97684	$2c_0-c_2$
$C=A-B$	−·07947	
$D=\frac{1}{3}C=c_2$	−·02649	
$2c_0=B+D$	1·95035	
$A'=F_0-F_0'$	−·63245	$2c_1+2c_3$
$B'=F_6-F_6'$	−·30353	c_1-2c_3
$3c_1=A'+B'$	−·93598	
c_1	−·31199	
$2c_3=c_1-B'$	−·00846	
$c_0\ (+c_6)$	·97518	
$c_1\ (+c_5)$	−·31199	
$c_2\ (+c_4)$	−·02649	
$c_3\ (+c_9)$	−·00423	

Addition for extra value $x=90°$

$\frac{1}{2}A$	·94869
F_9	1·00000
$\frac{1}{2}A-F_9=2c_2$	−·05131
c_2	−·02566
$D-c_2=c_4$	−·00083
$c_0\ (+c_6)$	·97518
$c_1\ (+c_5)$	−·31199
$c_2\ (+c_6)$	−·02566
$c_3\ (+c_9)$	−·00423
$c_4\ (-c_6)$	−·00083

A·8. Schedule for 3 Harmonics. Sine Series.

Values of $x=60°,\ 120°,\ 90°$. $F=\cdot 3\sin x\,(1-\cdot 6\cos x)^{-\frac{1}{2}}$.

F_6	·31053
F_6'	·22787
F_9	·30000
$s_1\sqrt{3}=F_6+F_6'$	·53840
s_1	·31085
$s_3=s_1-F_9$	·01085
$s_2\sqrt{3}=F_6-F_6'$	·08266
s_2	·04772

$s_1\ (-s_5)$	·31085
$s_2\ (-s_4)$	·04772
$s_3\ (-2s_5)$	·01085

A·9. Schedule for 4 Harmonics. Cosine Series.

Values of $x = 0°, 180°, 90°, 45°, 135°$. $F = (1 - ·6 \cos x)^{\frac{1}{2}}$.

F_0	·63246	
F_0'	1·26491	
F_4	·75877	
F_4'	1·19343	
F_9	1·00000	
$A = F_0 + F_0'$	1·89737	$2c_0 + 2c_2 + 2c_4$
$B = \frac{1}{2} A$	·94869	
$2c_2 = B - F_9$	− ·05131	
$C = B + F_9$	1·94869	$2c_0 + 2c_4$
$D = F_4 + F_4'$	1·95220	$2c_0 - 2c_4$
$4c_0 = C + D$	3·90089	
$4c_4 = C - D$	− ·00351	

$A' = F_4 - F_4'$	− ·43466	$\sqrt{2}\,(c_1 - c_3)$
$B' = \sqrt{2}\, A'$	− ·61470	$2c_1 - 2c_3$
$C' = F_0 - F_0'$	− ·63245	$2c_1 + 2c_3$
$4c_3 = C' - B'$	− ·01775	
$4c_1 = C' + B'$	− 1·24715	
$c_0\ (+c_8)$	·97522	
$c_1\ (+c_7)$	− ·31179	
$c_2\ (+c_6)$	− ·02566	
$c_3\ (+c_5)$	− ·00444	
$c_4\ (+c_{12})$	− ·00088	

A·10. Schedule for 4 or 5 Harmonics. Sine Series.

Values of $x = 0°, 180°, 90°, 45°, 135°$. $F = ·3 \sin x\,(1 - ·6 \cos x)^{-\frac{1}{2}}$.

F_d	·47434	
F_d'	·23717	
F_9	·30000	
F_4	·27957	
F_4'	·17775	
$A = F_4 + F_4'$	·45732	$\sqrt{2}\,(s_1 + s_3 - s_5)$
$B = A \div \sqrt{2}$	·32337	$s_1 + s_3 - s_5$
$2s_1 = B + F_9$	·62337	
$C = B - F_9$	·02337	$2s_3 - 2s_5$
$D = F_d + F_d'$	·71151	$2s_1 + 6s_3 + 10s_5$
$2F_9$	·60000	
$E = D - 2F_9$	·11151	$8s_3 + 8s_5$
$G = \frac{1}{4} E$	·02788	$2s_3 + 2s_5$
$4s_3 = G + C$	·05125	
$4s_5 = G - C$	·00451	

$A' = F_d - F_d'$	·23717	$4s_2 + 8s_4$
$B' = \frac{1}{2} A'$	·11859	$2s_2 + 4s_4$
$2s_2 = F_4 - F_4'$	·10182	
$4s_4 = B' - 2s_2$	·01677	
$s_1\ (-s_7)$	·31169	
$s_2\ (-s_6)$	·05091	
$s_3\ (+s_7)$	·01281	
$s_4\ (-2s_6)$	·00419	
$s_5\ (+s_7)$	·00113	

A·11. Schedule for 6 Harmonics. Cosine Series.

Values of $x=0°$, $180°$, $90°$, $30°$, $150°$, $60°$, $120°$. $F=(1-\cdot 6\cos x)^{\frac{1}{2}}$.

F_0	·63246	
F_0'	1·26491	
F_9	1·00000	
F_3	·69311	
F_3'	1·23272	
F_6	·83666	
F_6'	1·14018	
$A=F_0+F_0'$	1·89737	$2c_0+2c_2+2c_4+2c_6$
$B=2F_9$	2·00000	$2c_0-2c_2+2c_4-2c_6$
$C=F_3+F_3'$	1·92583	$2c_0+c_2-c_4-2c_6$
$D=F_6+F_6'$	1·97684	$2c_0-c_2-c_4+2c_6$
$E=A+B$	3·89737	$4c_0+4c_4$
$F=C+D$	3·90267	$4c_0-2c_4$
$6c_4=E-F$	−·00530	
$2c_4$	−·00177	
$4c_0=F+2c_4$	3·90090	
$G=A-B$	−·10263	$4c_2+4c_6$
$H=C-D$	−·05101	$2c_2-4c_6$
$6c_2=G+H$	−·15364	
$2c_2$	−·05121	
$4c_6=2c_2-H$	−·00020	

$A'=F_0-F_0'$	−·63245	$2c_1+2c_3+2c_5$
$B'=F_6-F_6'$	−·30352	$c_1-2c_3+c_5$
$C'=F_3-F_3'$	−·53961	$\sqrt{3}(c_1-c_5)$
$D'=A'+B'$	−·93597	$3c_1+3c_5$
$E'=\sqrt{3}\,C'$	−·93463	$3c_1-3c_5$
$6c_1=D'+E'$	−1·87060	
$6c_5=D'-E'$	−·00134	
$G'=\frac{1}{3}D'$	−·31199	c_1+c_5
$2c_3=G'-B'$	−·00847	
$c_0\,(+c_{12})$	·97523	
$c_1\,(+c_{11})$	−·31177	
$c_2\,(+c_{10})$	−·02561	
$c_3\,(+c_9)$	−·00423	
$c_4\,(+c_8)$	−·00089	
$c_5\,(+c_7)$	−·00022	
$c_6\,(+c_{13})$	−·00005	

A·12. Schedule for 6 or 7 Harmonics. Sine Series.

Values of $x = 0°, 180°, 90°, 30°, 150°, 60°, 120°$. $F = {\cdot}3 \sin x\,(1 - {\cdot}6 \cos x)^{-\frac{1}{2}}$.

F_d	·47434	
F_d'	·23717	
F_9	·30000	
F_3	·21642	
F_3'	·12168	
F_6	·31053	
F_6'	·22787	
$F_3 + F_3'$	·33810	$s_1 + 2s_3 + s_5 - s_7$
$3s_3 = F_3 + F_3' - F_9$	·03810	
s_3	·01270	
$A = F_6 + F_6'$	·53840	$\sqrt{3}\,(s_1 - s_5 + s_7)$
$B = A \div \sqrt{3}$	·31085	$s_1 - s_5 + s_7$
$C = s_3 + F_9$	·31270	$s_1 + s_5 - s_7$
$2s_1 = B + C$	·62355	
$D = C - B$	·00185	$2s_5 - 2s_7$
$E = F_d + F_d'$	·71151	$2s_1 + 6s_3 + 10s_5 + 14s_7$
$G = E - 2s_1$	·08796	$6s_3 + 10s_5 + 14s_7$
$H = -6s_3$	$-$·07620	$-6s_3$
$K = -5D$	$-$·00925	$-10s_5 + 10s_7$
$24s_7 = G + H + K$	·00251	
$2s_7$	·00021	
$2s_5 = D + 2s_7$	·00206	

$A' = F_3 - F_3'$	·09474	$\sqrt{3}\,(s_2 + s_4)$
$B' = F_6 - F_6'$	·08266	$\sqrt{3}\,(s_2 - s_4)$
$C' = 2A' \div \sqrt{3}$	·10940	$2s_2 + 2s_4$
$D' = 2B' \div \sqrt{3}$	·09545	$2s_2 - 2s_4$
$4s_4 = C' - D'$	·01395	
$4s_2 = C' + D'$	·20485	
$E' = F_d - F_d'$	·23717	$4s_2 + 8s_4 + 12s_6$
$G' = E' - 4s_2$	·03232	$8s_4 + 12s_6$
$8s_4$	·02790	
$12s_6 = G' - 8s_4$	·00442	

$s_1\,(-s_{11})$	·31178
$s_2\,(-s_{10})$	·05121
$s_3\,(-s_9)$	·01270
$s_4\,(-s_8)$	·00349
$s_5\,(+s_9)$	·00103
$s_6\,(+2s_8)$	·00037
$s_7\,(+s_9)$	·00010

A·13. *Double Harmonic Analysis.* When a function of two angles can be expressed in the form

$$F(x, y) = \Sigma\,(A_{j,j'} \cos jx \cos j'y + B_{j,j'} \sin jx \sin j'y$$
$$+ C_{j,j'} \cos jx \sin j'y + D_{j,j'} \sin jx \cos j'y),$$

where $j, j' = 0, 1, 2, \ldots$, the numerical calculation of the coefficients can be reduced to a double application of single harmonic analyses.

First, the choice of pairs of values α, $2\pi - \alpha$ separates the terms containing cosines of multiples of x from those containing sines by addition and subtraction. A similar choice for y makes a similar separation in each case. Thus the analysis is reduced to that of each of the four groups, and values for x, y equal to or less than 180° are to be used.

A·14. Consider the first group given by

$$4A_{x,y} = F(x, y) + F(2\pi - x, y) + F(x, 2\pi - y)$$
$$+ F(2\pi - x, 2\pi - y)$$
$$= 4\Sigma A_{j,j'} \cos jx \cos j'y.$$

The special values of $A_{x,y}$ are arranged in a block, each line containing those corresponding to a special value of x, and each column those corresponding to a special value of y.

The special values in each column are analysed by one of the schedules for cosine analysis and give series $\Sigma A_{j,y} \cos jx$, in each of which y has a special value.

The results are rearranged in a block in which all the numbers $A_{j,y}$ corresponding to a given j are placed in a column, the successive columns thus containing the special values of the coefficients of $\cos 0x$, $\cos x$, $\cos 2x$, ... for the special values of y.

The special values in each column are then analysed by one of the schedules for cosine analysis and give the coefficients $A_{j,j'}$.

The process for finding the $B_{j,j'}$ is the same with the exception that the cosine analyses are replaced by sine analyses.

For $C_{j,j'}$, the first block of analyses is that for cosines, and the second that for sines.

For $D_{j,j'}$, the first block of analyses is that for sines, and the second that for cosines.

The choices of special values of x, y are made on the same plans as those developed for single harmonic analysis. It is not necessary that the same choices of values be adopted for x as for y.

A·15. The derivatives

$$\left(\frac{\partial F}{\partial x}\right)_{x=0}, \quad -\left(\frac{\partial F}{\partial x}\right)_{x=\pi}, \quad \left(\frac{\partial F}{\partial y}\right)_{y=0}, \quad -\left(\frac{\partial F}{\partial y}\right)_{y=\pi}$$

are to be used in the sine analyses instead of the zero values of the functions F. This is possible because the derivatives of the $\cos jx$ terms in F disappear from $\partial F/\partial x$ when $x = 0, \pi$; and similarly those of the $\cos j'y$ terms from $\partial F/\partial y$ when $y = 0, \pi$.

A·16. In the method of development of the disturbing function and disturbing forces outlined in 4·15, the separation into the four sets of terms is made at the outset. Each function which vanishes for $g = 0, \pi$ or for $g' = 0, \pi$, is replaced by a derivative as shown above.

When the development is made in terms of the angles f, f_1 (4·19), the special values of the first two sets are found together and must be separated by addition and subtraction: similarly for the third and fourth sets.

An example in which the calculations are shown in detail will be found in the *Tables for the Development of The Disturbing Function*, by Brown and Brouwer*.

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